

COFIBRANTLY GENERATED LAX ORTHOGONAL FACTORISATION SYSTEMS

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ABSTRACT. The present note has three aims. First, to complement some of the theory of cofibrant generation of algebraic weak factorisation systems (AWFSS) to cover some important examples that are not locally presentable categories. Secondly, to prove that cofibrantly KZ-generated AWFSS (a notion we define) are always lax idempotent (in the sense of Clementino and the author). Thirdly, to show that the two known methods of building lax orthogonal AWFSS, namely cofibrantly KZ-generation and the method of “simple adjunctions”, construct different AWFSS. We do so by exhibiting two examples: an AWFSS on the category of preorders whose fibrant objects are complete lattices, and one on the category of topological spaces whose fibrant objects are continuous lattices.

1. INTRODUCTION

Various aspects of cofibrant generation of algebraic weak factorisation systems – henceforth abbreviated AWFSS – on locally presentable categories were studied in [4], encompassing a large family of examples that, however, do not reach some important ones, as those based on the category of topological spaces. The present note is an attempt to fill this gap in the literature. The article can be divided in to parts, the first, encompassing most of the article, that deals with cofibrant generation of AWFSS enriched over a base category $\mathcal{V} \subseteq \mathbf{Cat}$, and the second that looks at the case when \mathcal{V} is the category of preorders.

AWFSS were introduced by M. Grandis and W. Tholen in [12], with latter contributions by R. Garner [11], and, as the name indicates, they are an algebraisation of the more classical notion of a weak factorisation system – abbreviated WFSS. A WFS consists of two classes of morphisms $(\mathcal{L}, \mathcal{R})$ with the property that each morphism f can be written as $f = r \cdot \ell$, with $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$, and each $r \in \mathcal{R}$ has the right lifting property with respect to each $\ell \in \mathcal{L}$: for each commutative square as displayed, there exists a – non necessarily unique – diagonal filler.

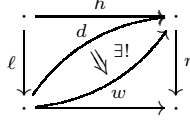
$$\begin{array}{ccc}
 & \xrightarrow{\quad} & \\
 \ell \downarrow & \nearrow & \downarrow r \\
 & \xrightarrow{\quad} &
 \end{array}
 \tag{1}$$

This kind of lifting situation was common in algebraic topology long before the importance of the concept of WFS was fully realised, for which Quillen’s definition of model category played a central role.

One of the features that distinguishes AWFSS from WFSS – on a category \mathcal{C} , say – is that the factorisation of morphisms is functorial – as is also the case in many WFSS constructed by the so-called Quillen’s small object argument [21]. The left and right classes of morphisms are replaced, respectively, by the coalgebras and algebras for a certain comonad and monad on \mathcal{C}^2 , and this extra algebraic structure ensures that diagonal fillers (1) not only exist but can be canonically constructed.

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Lax orthogonal AWFSS, introduced in [6], are AWFSS on 2-categories for which the canonical diagonal, say d , satisfy a certain universal property with respect to 2-cells: given any other diagonal filler w , there exists a unique 2-cell $\alpha: d \Rightarrow w$ such that $\alpha \cdot \ell = 1$ and $r \cdot \alpha = 1$.



See Section 4 or [6] for a precise statement, along with the basic theory of lax orthogonal AWFSS and a procedure to construct them via the so-called *simple adjunctions*. For example, when the codomain of r is a terminal object, the above condition says that the identity 2-cell $h = d \cdot \ell$ exhibits d as a left Kan extension of h along ℓ . Objects A with this property with respect to a family of morphisms ℓ have been studied in the context of poset-enriched categories and called *Kan objects* [9] or *Kan injective objects* [2].

The notion of cofibrant generation adapted to AWFSS was introduced in [11], and latter extended to generation by a double category in [4], where, in addition, enriched cofibrant generation is discussed.

There is a notion of cofibrant generation for lax orthogonal AWFSS, that we call *cofibrant KZ-generation*, and we show that cofibrantly KZ-generated AWFSS are always lax idempotent. Cofibrant KZ-generation can be seen as a case of the constructions in [4, §8] – even though [4] concentrates on locally presentable categories. There are important examples, however, that are not locally presentable categories, as the category of topological spaces. In Section 7 we study the case of categories enriched in preorders and the cofibrant KZ-generation thereon, encompassing in this way the example of topological spaces.

We have mentioned two ways of constructing new lax orthogonal AWFSS: via simple adjunctions and via cofibrant KZ-generation. It is natural to ask whether these two ways construct the same AWFSS; we give a negative answer to this question in Section 8. We exhibit two lax orthogonal AWFSS that can be constructed via simple adjunctions but are not cofibrantly KZ-generated, nor cofibrantly generated; furthermore, their underlying WFSs are not cofibrantly generated, in the usual sense of the term. The two AWFSS in question can be described as follows. The first of these is the AWFSS on the category of preorders whose left morphisms are the full morphisms of preorders and whose fibrant objects are the complete lattices. The second example is the AWFSS on the category of T_0 topological spaces whose left morphisms are the subspace embeddings and whose fibrant objects are the continuous lattices – [6, §12].

We conclude this introduction with a description of the article contents. Section 2 collects some of the constructions and results relative to AWFSS needed in the rest of the article. We adapt in Section 3 the notions of cofibrant generation introduced in [11, 4] to the case of categories enriched in $\mathcal{V} \subseteq \mathbf{Cat}$. Lax idempotent 2-monads and lax orthogonal AWFSS and the notion of cofibrant KZ-generation are recalled in Section 4, and we go on to show in Section 5 that cofibrantly KZ-generated AWFSS are lax orthogonal. Section 6 shows that cofibrantly generated and KZ-generated AWFSS must satisfy certain colimit creation property, to be used in a later section. Section 7 proves an existence result for cofibrantly KZ-generated AWFSS on preorder-enriched categories. Section 8 is divided in two parts, each exhibiting a lax orthogonal AWFSS that can be constructed by the method of simple adjunctions [6] but is *not* cofibrantly KZ-generated, nor cofibrantly generated, and whose underlying WFS is *not* cofibrantly generated in the usual meaning of the term. There is a short appendix comparing the present note with the article [2].

2. BACKGROUND ON ALGEBRAIC WEAK FACTORISATION SYSTEMS

As mentioned in the introduction, we are interested in 2-categories and locally preordered categories. A possible path one can choose in the exposition is to work with \mathcal{V} -enriched categories, for a fairly general symmetric monoidal closed category \mathcal{V} . On the other hand, one could treat only the case of 2-categories and, when necessary, argue that the relevant constructions and results restrict to locally preordered 2-categories. We will take a middle of the road approach and consider categories enriched in a full sub-2-category $\mathcal{V} \subseteq \mathbf{Cat}$, closed under limits, colimits and exponentials. Furthermore, it will be important for our applications that the arrow category $\mathbf{2}$ should belong to \mathcal{V} . We have in mind the 2-categories \mathbf{Cat} of small categories, \mathbf{Pord} of preorders and \mathbf{Pos} of posets. Many of the \mathcal{V} -enriched notions we discuss below hold for a general \mathcal{V} , and are not difficult to elaborate in that context, but since they do not add much to our main examples, are left to be developed in another place.

Each \mathcal{V} -category has an underlying 2-category, and conversely, a 2-category is of this form if and only if its hom-categories lie in \mathcal{V} . We refer to the 2-cells of the underlying 2-category of \mathcal{A} simply as 2-cells of \mathcal{A} .

2.1. Functorial factorisations. A \mathcal{V} -functorial factorisation on a \mathcal{V} -category \mathcal{C} is a \mathcal{V} -functor $\mathcal{C}^2 \rightarrow \mathcal{C}^3$ that is a section of the composition \mathcal{V} -functor $\mathcal{C}^3 \rightarrow \mathcal{C}^2$. Equivalently, it is a \mathcal{V} -functor $K: \mathcal{C}^2 \rightarrow \mathcal{C}$ with \mathcal{V} -natural transformations $\text{dom} \Rightarrow K \Rightarrow \text{cod}$ whose composition equals the canonical transformation $\text{dom} \Rightarrow \text{cod}$ with f -component equal to f . In other words, a \mathcal{V} -functorial factorisation is a functorial factorisation as defined in [12] that is compatible with the 2-cells of \mathcal{C} .

$$(A \xrightarrow{f} B) = (A \xrightarrow{\lambda_f} Kf \xrightarrow{\rho_f} B)$$

As in the case of ordinary categories, a functorial factorisation can be equivalently described by a copointed endo- \mathcal{V} -functor $\Phi: L \Rightarrow 1$ on \mathcal{C}^2 with $\text{dom } \Phi = 1$, and by a pointed endo- \mathcal{V} -functor $\Lambda: 1 \Rightarrow R$ on \mathcal{C}^2 with $\text{cod } \Lambda = 1$. We sometimes write $Lf = \lambda_f$ and $Rf = \rho_f$.

Given a \mathcal{V} -functorial factorisation as in the previous paragraph, each coalgebra structure $(1, s): f \rightarrow Lf$ for the copointed endo- \mathcal{V} -functor (L, Φ) on $f \in \mathcal{C}^2$ and each algebra structure $(p, 1): Rf \rightarrow f$ for the pointed endo- \mathcal{V} -functor (R, Λ) on $g \in \mathcal{C}^2$ induces a choice of diagonal fillers for morphisms $(h, k): f \rightarrow g$ in \mathcal{C}^2 , ie commutative squares in \mathcal{C} . The diagonal filler for this square is the composite

$$\text{diag}(h, k): \text{cod } f = \text{dom } k \xrightarrow{s} Kf \xrightarrow{K(h, k)} Kg \xrightarrow{p} \text{cod } h = \text{dom } g.$$

If $(\alpha, \beta): (h, k) \Rightarrow (\bar{h}, \bar{k}): f \rightarrow g$ is a 2-cell in \mathcal{C}^2 , then there is a corresponding 2-cell $\text{diag}(\alpha, \beta): \text{diag}(h, k) \Rightarrow \text{diag}(\bar{h}, \bar{k})$, given by $\text{diag}(\alpha, \beta) = p \cdot K(\alpha, \beta) \cdot s$. See [12, 11] for details.

2.2. Enriched AWFSS. There is a close relationship between AWFSS and double categories, as investigated in [4]. We begin by considering the latter. Recall that the cocategory structure on $\mathbf{2}$ induces an internal category $\text{Sq}(\mathcal{C})$ in $\mathcal{V}\text{-}\mathbf{CAT}$, depicted on the right, and called the *internal category of squares* in \mathcal{C} . When \mathcal{V} is the category of sets, internal categories in \mathbf{CAT} are usually called *double categories*.

$$\mathbf{3} \begin{array}{c} \longleftarrow \\ \longleftarrow \\ \longleftarrow \end{array} \mathbf{2} \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} \mathbf{1} \quad \mathcal{C}^3 \begin{array}{c} \longleftarrow \\ \longleftarrow \\ \longleftarrow \end{array} \mathcal{C}^2 \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} \mathcal{C}$$

A \mathcal{V} -enriched AWFSS on a \mathcal{V} -category \mathcal{C} is a \mathcal{V} -functorial factorisation equipped with a comultiplication $\Sigma: L \Rightarrow L^2$ that makes $L = (L, \Phi, \Sigma)$ a \mathcal{V} -enriched comonad, and a multiplication $\Pi: R^2 \Rightarrow R$ that makes $R = (R, \Lambda, \Pi)$ a \mathcal{V} -enriched monad. Furthermore, it is required that the underlying ordinary comonad and monad on the underlying category of \mathcal{C}^2 should form an AWFSS as defined in [11]; in other

words, the \mathcal{V} -natural transformation $LR \Rightarrow RL$ with components $(\sigma_f, \pi_f): LRf \rightarrow RLf$ must be a mixed distributive law; here we have used the notation $\Sigma_f = (1, \sigma_f): Lf \rightarrow L^2f$ and $\Pi_f = (\pi_f, 1): R^2f \rightarrow Rf$. This distributivity condition, added in [11] to the original definition of AWFS – called natural WFSs in [12] – is precisely what is needed in order to have an associative composition of R-algebras and of L-coalgebras. An associative composition of R-algebras chooses, for each pair of R-algebras f, g with $\text{cod } f = \text{dom } g$, an R-algebra structure on the composition $g \cdot f$; these assignments must be natural with respect to morphisms and 2-cells of R-algebras, and it must be associative in the sense that the R-algebra structures of $(h \cdot g) \cdot f$ and $h \cdot (g \cdot f)$ must coincide. A similar statement can be made about L-coalgebras.

The existence of the associative composition mentioned in the previous paragraph can be rephrased in a more concise way: if (L, R) is a \mathcal{V} -enriched AWFS, there are internal categories in $\mathcal{V}\text{-CAT}$

$$\text{L-Coalg} \rightleftarrows \mathcal{C} \quad \text{R-Alg} \rightleftarrows \mathcal{C}$$

that we denote by L-Coalg and R-Alg . These two internal categories come equipped with internal functors into $\text{Sq}(\mathcal{C})$ given by forgetting the (co)algebra structure. Furthermore, given a \mathcal{V} -monad R on \mathcal{C} , there is a bijection between compositions that make $\text{R-Alg} \rightrightarrows \mathcal{C}$ into an internal category in $\mathcal{V}\text{-CAT}$ and \mathcal{V} -comonads L such that (L, R) is a \mathcal{V} -enriched AWFS. This is completely analogous to the case of ordinary AWFSs due to R. Garner – see, for example, [4, §2.8].

The internal categories in $\mathcal{V}\text{-CAT}$ that arise from an AWFS can be characterised as done in [4, §3]. If $\mathbb{D} = (\mathcal{D}_1 \rightrightarrows \mathcal{D}_0)$ is an internal category and $U: \mathbb{D} \rightarrow \text{Sq}(\mathcal{C})$ an internal functor in $\mathcal{V}\text{-CAT}$, then \mathbb{D} is isomorphic to R-Alg over $\text{Sq}(\mathcal{C})$, for a – essentially unique – \mathcal{V} -enriched AWFS (L, R) on \mathcal{C} , if $U_1: \mathcal{D}_1 \rightarrow \mathcal{C}^2$ is strictly monadic and the induced \mathcal{V} -monad is isomorphic to a codomain-preserving \mathcal{V} -monad. A more elementary condition can be found in [4, Thm. 6].

2.3. Discrete pullback-fibrations of \mathcal{V} -categories. There is another fact about AWFSs that we shall use in Section 3. Given a codomain-preserving monad R on \mathcal{C}^2 , if $(h, k): f \rightarrow g$ is a morphism in \mathcal{C}^2 that is a pullback square and g carries an R-algebra structure, then there exists a unique R-algebra structure on f that makes (h, k) a morphism of R-algebras. In the terminology of [4, §3.4], $\text{R-Alg} \rightarrow \mathcal{C}^2$ is a *discrete pullback-fibration*.

Let $P: \mathcal{E} \rightarrow \mathcal{B}$ be a \mathcal{V} -functor and \mathcal{S} a family of morphisms in \mathcal{B} closed under composition and that satisfies: if $s \in \mathcal{S}$ and $s \cdot t \in \mathcal{S}$, then $t \in \mathcal{S}$ – this implies that $\{1_{\text{dom}(s)} : s \in \mathcal{S}\} \subset \mathcal{S}$. We say that P is a *discrete \mathcal{S} -fibration* if it satisfies:

- (1) Given $e \in \mathcal{E}$ and $s: b \rightarrow P(e)$ in \mathcal{S} there exists a unique $t: s^*(e) \rightarrow e$ such that $P(t) = s$;
- (2) Given two morphisms $u, v: e' \rightarrow e$ in \mathcal{E} such that $P(u), P(v) \in \mathcal{S}$, and a 2-cell $\beta: P(u) \Rightarrow P(v)$, there exists a unique $\tilde{\beta}: u \Rightarrow v$ such that $P(\tilde{\beta}) = \beta$.

For a codomain-preserving \mathcal{V} -monad R on \mathcal{C}^2 , the forgetful \mathcal{V} -functor $\text{R-Alg} \rightarrow \mathcal{C}^2$ is a discrete \mathcal{S} -fibration for \mathcal{S} the class of morphisms in \mathcal{C}^2 that are pullback squares. The verification of the condition (1) is identical to [4, Prop. 8], while condition (2) is similarly proven.

2.4. Internal categories of LARIS and RALIS. Later we will need to refer to certain internal \mathcal{V} -categories whose vertical morphism are given by adjunctions.

A morphism $f: A \rightarrow B$ in a \mathcal{V} -category equipped with a right adjoint whose unit is an identity 2-cell may be called a *left adjoint right inverse*, abbreviated LARI, following terminology used in [13, 4]. A morphism of LARIS is a morphism between the underlying morphisms in \mathcal{C} that commutes with the right adjoints and

the corresponding counits. For any \mathcal{V} -category \mathcal{A} there is a \mathcal{V} -category $\mathbf{Lari}(\mathcal{A})$; clearly, LARIS compose, so there is an internal category $\mathbf{Lari}(\mathcal{A}) \rightrightarrows \mathcal{A}$ in $\mathcal{V}\text{-CAT}$, denoted by $\mathbb{Lari}(\mathcal{A})$, and an obvious internal functor $\mathbb{Lari}(\mathcal{A}) \rightarrow \mathbb{Sq}(\mathcal{A})$.

Dually, a morphism $f: A \rightarrow B$ in a \mathcal{V} -category \mathcal{A} equipped with a left adjoint whose unit is an identity may be called a *right adjoint left inverse*, or RALI. A morphism of RALIS is a morphism in \mathcal{A}^2 between the underlying morphisms that commutes with the left adjoints and the units. There is a internal category $\mathbf{Rali}(\mathcal{A})$ in $\mathcal{V}\text{-CAT}$ and an forgetful internal functor $\mathbf{Rali}(\mathcal{A}) \rightarrow \mathbb{Sq}(\mathcal{A})$.

Each one of the internal categories $\mathbf{Lari}(\mathcal{A})$ and $\mathbf{Rali}(\mathcal{A})$ over $\mathbb{Sq}(\mathcal{A})$ are induced by an AWFS under weak assumptions on the \mathcal{V} -category \mathcal{A} . Recall that, as part of our assumptions, the category $\mathcal{V} \subseteq \mathbf{Cat}$ contains the arrow category $\mathbf{2}$ and is closed under limits so it makes sense to speak of comma-objects in \mathcal{V} ; these comma-objects are just comma-categories. The lax limit of a morphism $f: X \rightarrow Y$ in \mathcal{V} is another name for the comma-category f/Y . A *lax limit* of a morphism $f: A \rightarrow B$ in a \mathcal{V} -category \mathcal{A} is a diagram as the one depicted on the left, that is sent by each representable $\mathcal{A}(C, -)$ to a lax limit in \mathcal{V} . A *lax colimit* of f is a diagram as shown on the right that is sent by each $\mathcal{A}(-, C)$ to a lax limit in \mathcal{V} .

$$\begin{array}{ccc} & A & \\ f/B \swarrow & \downarrow f & \searrow \\ & B & \end{array} \quad \begin{array}{ccc} & A & \\ f \swarrow & & \searrow \text{col}_\ell f \\ B & \uparrow & \end{array}$$

If \mathcal{A} has lax limits of morphisms, then $\mathbb{Lari}(\mathcal{A}) \cong \mathbf{L}\text{-Coalg}$, where (\mathbf{L}, \mathbf{R}) are given on a morphism $f: A \rightarrow B$ in the following way: $L(f): A \rightarrow f/B$ is the morphism that corresponds to the identity 2-cell $f \Rightarrow f$, and $R(f): f/B \rightarrow B$ is the projection; see [6, §3.3].

If \mathcal{A} has lax colimits of morphisms, then there is a \mathcal{V} -enriched AWFS (\mathbf{E}, \mathbf{M}) on \mathcal{A} such that $\mathbf{Rali}(\mathcal{A}) \cong \mathbf{M}\text{-Alg}$, where (\mathbf{E}, \mathbf{M}) is given on a morphism $f: A \rightarrow B$ in the following way: $E(f): A \rightarrow \text{col}_\ell f$ is the coprojection and $M(f): \text{col}_\ell f \rightarrow B$ is the morphism corresponding to the identity 2-cell $f \Rightarrow f$.

Remark 1. Let (M, Λ^M) the pointed endo- \mathcal{V} -functor underlying the \mathcal{V} -monad \mathbf{M} of the previous paragraph. The \mathcal{V} -category of (M, Λ^M) -algebras has as objects morphisms $f: A \rightarrow B$ of \mathcal{C} equipped with a morphism $f^\ell: B \rightarrow A$ and a 2-cell $\varepsilon: f^\ell \cdot f \Rightarrow 1_A$ that satisfy $f \cdot f^\ell = 1$ and $f \cdot \varepsilon = 1$. This description will be used in later sections.

3. COFIBRANT GENERATION

There is a notion of cofibrant generation for AWFSs on enriched categories, considered in [4] – the AWFSs of [4] are not enriched, however – which in our setting can be described as follows. Suppose that \mathcal{V} is equipped with an AWFS (\mathbf{E}, \mathbf{M}) . For a \mathcal{V} -functor $U: \mathcal{J} \rightarrow \mathcal{C}^2$ with \mathcal{J} small, consider the \mathcal{V} -functor $W_U: \mathcal{C}^2 \rightarrow [\mathcal{J}^{\text{op}}, \mathcal{V}]^2$ into the \mathcal{V} -category of \mathcal{V} -presheaves given by

$$W_U(f) = \tilde{U}(1_{\text{dom } f}, f) = \mathcal{C}^2(U-, (1, f)): \mathcal{C}^2(U-, 1) \longrightarrow \mathcal{C}^2(U-, f)$$

where $(1, f): 1_{\text{dom}(f)} \rightarrow f$, for $f \in \mathcal{C}^2$, and define $\mathcal{J}^{\text{ch}_M}$ as the pullback

$$\begin{array}{ccc} \mathcal{J}^{\text{ch}_M} & \longrightarrow & \mathbf{M}\text{-Alg} \\ U^{\text{ch}_M} \downarrow & & \downarrow \\ \mathcal{C}^2 & \xrightarrow{W_U} & [\mathcal{J}^{\text{op}}, \mathcal{V}]^2 \end{array} \quad (2)$$

Here \mathbf{M} denotes the induced \mathcal{V} -monad on $[\mathcal{J}^{\text{op}}, \mathcal{V}]$ defined by pointwise application of the \mathcal{V} -monad of the same name on \mathcal{V}^2 . When \mathcal{J} is large, one can still define \mathcal{J}^{hM} as a **CAT**-category.

Remark 2. Observe that W_U preserves limits and, therefore, U^{hM} creates limits. Indeed, consider the \mathcal{V} -functor $\mathcal{C}^t: \mathcal{C}^2 \rightarrow \mathcal{C}^{2 \times 2} \cong (\mathcal{C}^2)^2$, where $t: 2 \times 2 \rightarrow 2$ is the functor that sends all objects to $0 \in 2$ except $t(1, 1) = 1 \in 2$; in other words $\mathcal{C}^t(f) = (1, f): 1_{\text{dom } f} \rightarrow f$. Then W_U is the composition of the obviously continuous \mathcal{C}^t with the continuous $\tilde{U}^2: (\mathcal{C}^2)^2 \rightarrow [\mathcal{J}^{\text{op}}, \mathcal{V}]^2$.

Furthermore, W_U has a left adjoint if \mathcal{C} is cocomplete and \mathcal{J} is small. To see this, observe that \mathcal{C}^t has a left adjoint given by left Kan extension, and \tilde{U} has a left adjoint given by $\varphi \mapsto \text{col}(\varphi, U)$ – the colimit of U weighted by φ .

The underlying category of \mathcal{J}^{hM} is part of a double category – see [4, §5.2]. Given two objects of \mathcal{J}^{hM} with composable underlying morphisms in \mathcal{C} , say $f: A \rightarrow B$ and $g: B \rightarrow C$, there is a composite object with underlying morphism $g \cdot f$. The \mathbf{M} -algebra structure on $W_U(g \cdot f)$ is described as follows. The functor \tilde{U} preserves limits, and, in particular, takes the pullback in \mathcal{C}^2 depicted on the left below to a pullback in $[\mathcal{J}^{\text{op}}, \mathcal{V}]$ depicted on the right.

$$\begin{array}{ccc}
 f & \xrightarrow{(f,1)} & 1_B \\
 (1,g) \downarrow & & \downarrow (1,g) \\
 g \cdot f & \xrightarrow{(f,1)} & g
 \end{array}
 \qquad
 \begin{array}{ccc}
 \tilde{U}(f) & \xrightarrow{\tilde{U}(f,1)} & \tilde{U}(1_B) \\
 \tilde{U}(1,g) \downarrow & & \downarrow \tilde{U}(1,g) \\
 \tilde{U}(g \cdot f) & \xrightarrow{\tilde{U}(f,1)} & \tilde{U}(g)
 \end{array}
 \tag{3}$$

Being the pullback of an \mathbf{M} -algebra, $\tilde{U}(1, g): \tilde{U}(f) \rightarrow \tilde{U}(g \cdot f)$ supports a canonical \mathbf{M} -algebra structure – Section 2.3 – so we can equip the composite morphism $\tilde{U}(1, g) \cdot \tilde{U}(1, f) = \tilde{U}(1, g \cdot f)$ with an \mathbf{M} -algebra structure. In this way we define a composition $g \bullet f$ in \mathcal{J}^{hM} that is routinely verified to be associative. Together with the obvious identity morphisms, we have a double category $(\mathcal{J}^{\text{hM}})_\circ$; more details can be found in [4, §6, §8].

A careful inspection of the description of the vertical composition of objects of \mathcal{J}^{hM} above the present lemma reveals that it is a \mathcal{V} -functor $\bullet: \mathcal{J}^{\text{hM}} \times_{\mathcal{C}} \mathcal{J}^{\text{hM}} \rightarrow \mathcal{J}^{\text{hM}}$. For a general base of enrichment \mathcal{V} this proof is more delicate than our, simpler case of $\mathcal{V} \subseteq \mathbf{Cat}$.

Lemma 3. *The vertical composition of the double category $(\mathcal{J}^{\text{hM}})_\circ$ underlies a \mathcal{V} -functor $\bullet: \mathcal{J}^{\text{hM}} \times_{\mathcal{C}} \mathcal{J}^{\text{hM}} \rightarrow \mathcal{J}^{\text{hM}}$.*

Proof. We only need to show that given 2-cells (α_0, α_1) and (α_1, α_2) in \mathcal{J}^{hM} as depicted in the first diagram in Figure 1, then (α_0, α_2) is a 2-cell in \mathcal{J}^{hM} too. In more explicit terms, we need to show that the pair of 2-cells $(\tilde{U}(\alpha_0, \alpha_1), \tilde{U}(\alpha_1, \alpha_2))$ shown in the bottom diagram of Figure 1 is of \mathbf{M} -algebras, where $\tilde{U}(1, g \cdot f)$ and $\tilde{U}(1, g' \cdot f')$ have the composition \mathbf{M} -algebra structure described in the previous paragraph.

We know, by hypothesis, that $(\tilde{U}(\alpha_0, \alpha_1), \tilde{U}(\alpha_1, \alpha_2))$ is a 2-cell in $\mathbf{M}\text{-Alg}$. It suffices to show that $(\tilde{U}(\alpha_0, \alpha_1), \tilde{U}(\alpha_0, \alpha_2))$ is a 2-cell in $\mathbf{M}\text{-Alg}$, since then, their vertical composition will be a 2-cell in $\mathbf{M}\text{-Alg}$ too. We will use that there is a pullback square as in the right hand side of (3) for g' and f' , namely

$$(\tilde{U}(f', 1), \tilde{U}(f', 1)): \tilde{U}(1, g') \longrightarrow \tilde{U}(1, g') \tag{4}$$

is a pullback; its domain and codomain are morphisms $\tilde{U}(f') \rightarrow \tilde{U}(g' \cdot f')$ and $\tilde{U}(1_{A'_1}) \rightarrow \tilde{U}(g')$. Since $\mathbf{M}\text{-Alg} \rightarrow [\mathcal{J}^{\text{op}}, \mathcal{V}]^2$ is a discrete pullback-fibration of \mathcal{V} -categories – Section 2.3 – it suffices to show that the composition of the 2-cell

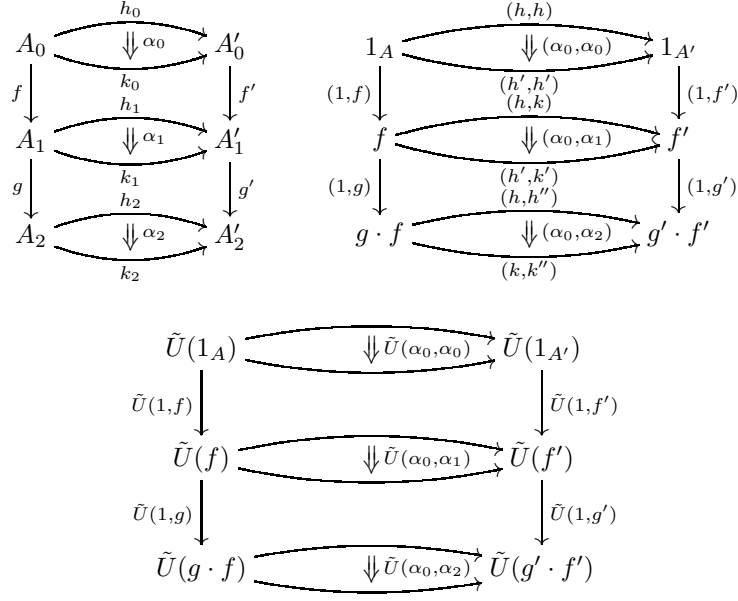


FIGURE 1.

$(\tilde{U}(\alpha_0, \alpha_1), \tilde{U}(\alpha_0, \alpha_2))$ with (4), ie

$$(\tilde{U}(f', 1) \cdot \tilde{U}(\alpha_0, \alpha_1), \tilde{U}(f', 1) \cdot \tilde{U}(\alpha_0, \alpha_2)) \quad (5)$$

is a 2-cell in $\mathbf{M}\text{-Alg}$. Calculating

$$\tilde{U}(f', 1) \cdot \tilde{U}(\alpha_0, \alpha_1) = \tilde{U}(\alpha_1, \alpha_1) \cdot \tilde{U}(f, 1)$$

$$\tilde{U}(f', 1) \cdot \tilde{U}(\alpha_0, \alpha_2) = \tilde{U}(\alpha_1, \alpha_2) \cdot \tilde{U}(f, 1)$$

and using that $\tilde{U}(\alpha_1, \alpha_1)$ and $\tilde{U}(\alpha_1, \alpha_2)$ are 2-cells of \mathbf{M} -algebras and $\tilde{U}(f, 1)$ is a morphism of \mathbf{M} -algebras, we deduce that (5) lies in $\mathbf{M}\text{-Alg}$, completing the proof. \square

Recall that a cotensor product of an object B in a \mathcal{V} -category \mathcal{C} by an object $V \in \mathcal{V}$ is an object $\{V, B\}$ together with a morphism $V \rightarrow \mathcal{C}(\{V, B\}, B)$ that induces isomorphisms $\mathcal{C}(A, \{V, B\}) \cong [V, \mathcal{C}(A, B)]$. For any \mathcal{V} -functor F there is a comparison morphism $F(\{V, X\}) \rightarrow \{V, F(X)\}$ induced by the universal property of cotensor products, if these two cotensor products exist.

Lemma 4. *Let \mathcal{C} be a \mathcal{V} -category with cotensor products with $\mathbf{2}$ and \mathcal{J} a small \mathcal{V} -category over \mathcal{C}^2 . The \mathcal{V} -category $\mathcal{J}^{\hat{\mathbf{M}}}$ is the object of arrows of an internal category in $\mathcal{V}\text{-}\mathbf{CAT}$, that we still denote by $\mathcal{J}^{\hat{\mathbf{M}}}$, and $U^{\hat{\mathbf{M}}}$ is the arrow part of an identity on objects internal functor $\mathcal{J}^{\hat{\mathbf{M}}} \rightarrow \mathbf{Sq}(\mathcal{C})$.*

Proof. We call a cotensor product with the object $\mathbf{2} \in \mathcal{V}$ a **2-cotensor**. Denote the underlying-category 2-functor from $\mathcal{V}\text{-}\mathbf{CAT}$ to \mathbf{CAT} by $(-)_\circ$, which is faithful when restricted to the 2-category of **2-cotensored** \mathcal{V} -categories and **2-cotensor preserving** \mathcal{V} -functors – because $\mathbf{2}$ is a (strong) generator in the ordinary category \mathbf{Cat} . Denote by U the given \mathcal{V} -functor from \mathcal{J} to \mathcal{C}^2 .

We know from [4, §8] that the result is true at the level of the underlying categories: the underlying category of $\mathcal{J}^{\hat{\mathbf{M}}}$ forms part of a double category and the underlying functor of $U^{\hat{\mathbf{M}}}$ forms part of an identity on objects double functor from $(\mathcal{J}^{\hat{\mathbf{M}}})_\circ$ to the double category of squares of \mathcal{C}_\circ . Therefore, in order to prove that $\mathcal{J}^{\hat{\mathbf{M}}}$ is an internal category in $\mathcal{V}\text{-}\mathbf{CAT}$ it will be enough to show that: $\mathcal{J}^{\hat{\mathbf{M}}}$ has

cotensor products with $\mathbf{2}$; and, the composition \mathcal{V} -functors of the internal category structure of $\mathcal{J}^{\mathfrak{h}\mathfrak{M}}$ preserves them. The first of these conditions follows from Remark 2.

Let f and g be two composable objects of $\mathcal{J}^{\mathfrak{h}\mathfrak{M}}$ and \bullet the vertical composition of the double category $(\mathcal{J}^{\mathfrak{h}\mathfrak{M}})_\circ$. The comparison morphism $\{\mathbf{2}, g\} \bullet \{\mathbf{2}, f\} \rightarrow \{\mathbf{2}, g \bullet f\}$ is defined by its projections into \mathcal{C}^2 and $\mathbf{M}\text{-Alg}$. The first is the identity $\{\mathbf{2}, g\} \bullet \{\mathbf{2}, f\} = \{\mathbf{2}, g \bullet f\}$ that expresses the fact that $\{\mathbf{2}, -\}$ is an endofunctor of \mathcal{C} . The second is the comparison morphism $\tilde{U}(\{\mathbf{2}, (1, g)\}) \cdot \tilde{U}(\{\mathbf{2}, (1, f)\}) \rightarrow \{\mathbf{2}, \tilde{U}(1, g \bullet f)\}$, which is an isomorphism too, as \tilde{U} preserves limits. \square

We now briefly discuss the construction $\mathbb{J}^{\mathfrak{h}\mathfrak{M}}$ of [4, §6]. Assume that \mathbb{J} is an internal category $\mathcal{J} \rightrightarrows \mathcal{J}_0$ in $\mathcal{V}\text{-Cat}$ and $U: \mathbb{J} \rightarrow \mathbf{Sq}(\mathcal{C})$ is an internal functor. First observe that, for vertically composable morphisms i and j in \mathbb{J} , the following square in \mathcal{V} is a pullback, where the top, right, left and bottom morphisms are, respectively, given by $(h, k) \mapsto (h \cdot Ui, h)$, $(h, k) \mapsto (h, g \cdot k)$, $(h, k) \mapsto (h \cdot Ui, k)$ and $(h, k) \mapsto (h, k \cdot Uj)$.

$$\begin{array}{ccc} \mathcal{C}^2(Uj, g) & \longrightarrow & \mathcal{C}^2(Ui, 1_{\text{dom}(g)}) \\ \downarrow & & \downarrow \tilde{U}(1, g)(i) \\ \mathcal{C}^2(Uj \cdot Ui, g) & \longrightarrow & \mathcal{C}^2(Ui, g) \end{array}$$

Since the morphism $\tilde{U}(1, g)(i)$ is an \mathbf{M} -algebra in \mathcal{V} , so is the morphism $\tilde{U}(g)(j) \rightarrow \tilde{U}(g)(j \bullet i)$ on the left of the pullback square, by Section 2.3; here $j \bullet i$ denotes the composition $\bullet: \mathcal{J} \times_{\mathcal{J}_0} \mathcal{J} \rightarrow \mathcal{J}$. Composing with the \mathbf{M} -algebra

$$\tilde{U}(1, g)(j): \tilde{U}(1)(j \bullet i) = \tilde{U}(1)(j) \longrightarrow \tilde{U}(g)(j)$$

we obtain an \mathbf{M} -algebra $\tilde{U}(1)(j) \rightarrow \tilde{U}(g)(j \bullet i)$. This construction is \mathcal{V} -natural in i and j , so we have a \mathcal{V} -functor $\mathcal{J}^{\mathfrak{h}\mathfrak{M}} \rightarrow (\mathcal{J} \times_{\mathcal{J}_0} \mathcal{J})^{\mathfrak{h}\mathfrak{M}}$, whose equaliser with the \mathcal{V} -functor induced by \bullet – ie, that which sends $i \mapsto \tilde{U}(g)(i)$ to $(j, i) \mapsto \tilde{U}(g)(j \bullet i)$ – we call $\mathbb{J}^{\mathfrak{h}\mathfrak{M}}$.

$$\mathbb{J}^{\mathfrak{h}\mathfrak{M}} \longrightarrow \mathcal{J}^{\mathfrak{h}\mathfrak{M}} \rightrightarrows (\mathcal{J} \times_{\mathcal{J}_0} \mathcal{J})^{\mathfrak{h}\mathfrak{M}} \quad (6)$$

There is an inclusion \mathcal{V} -functor $\mathbb{J}^{\mathfrak{h}\mathfrak{M}} \hookrightarrow \mathcal{J}^{\mathfrak{h}\mathfrak{M}}$ and $\mathbb{J}^{\mathfrak{h}\mathfrak{M}}$ is closed under composition in the internal category $\mathcal{J}^{\mathfrak{h}\mathfrak{M}} \rightrightarrows \mathcal{C}$, so we obtain an internal category $\mathbb{J}^{\mathfrak{h}\mathfrak{M}} \rightrightarrows \mathcal{C}$.

The inclusion $\mathbb{J}^{\mathfrak{h}\mathfrak{M}} \hookrightarrow \mathcal{J}^{\mathfrak{h}\mathfrak{M}}$ is full and faithful. To see this, observe that the two parallel \mathcal{V} -functors of (6) commute with the faithful functors into \mathcal{C}^2 ; details are left to the reader.

A \mathcal{V} -enriched AWFS (\mathbf{L}, \mathbf{R}) on \mathcal{C} is *cofibrantly generated* by \mathbb{J} when there is an isomorphism of internal categories $\mathbf{R}\text{-Alg} \cong \mathbb{J}^{\mathfrak{h}\mathfrak{M}}$ over $\mathbf{Sq}(\mathcal{C})$; see [4, §6.2]. Any two AWFSs cofibrantly generated by the same \mathbb{J} are isomorphic, so we may speak of *the* AWFS cofibrantly generated by \mathbb{J} . The \mathcal{V} -enriched AWFS cofibrantly generated by an internal category \mathbb{J} in $\mathcal{V}\text{-Cat}$ exists if and only if $\mathbb{J}^{\mathfrak{h}\mathfrak{M}} \rightarrow \mathcal{C}^2$ is strictly monadic. This can be shown by modifying [4, Thm. 6] to the case of \mathcal{V} -categories. When $\mathbb{J} = (\mathcal{J} \rightrightarrows \mathcal{J}_0)$ is large one can still define a **CAT**-category $\mathbb{J}^{\mathfrak{h}\mathfrak{M}}$. We say that (\mathbf{L}, \mathbf{R}) is cofibrantly generated *by a \mathcal{V} -category \mathcal{J}* over \mathcal{C}^2 if there is an isomorphism of internal categories $\mathbf{R}\text{-Alg} \cong \mathcal{J}^{\mathfrak{h}\mathfrak{M}}$ over $\mathbf{Sq}(\mathcal{C})$. We will see in Lemma 8 that, instead of giving a different notion of cofibrant generation, generation by a \mathcal{V} -category can be seen as a particular instance of cofibrant generation by an internal category.

Notation 5. When we consider cofibrant generation with respect to the AWFS (\mathbf{E}, \mathbf{M}) on \mathcal{V} whose \mathbf{M} -algebras are split epimorphisms, we will simply write $\mathbb{J}^{\mathfrak{h}}$ and $\mathcal{J}^{\mathfrak{h}}$, leaving \mathbf{M} understood. This notation coincides with that of [11, 4].

There is a simple description of objects of \mathcal{J}^\flat , first given in [11]; and object (g, φ) consists of a morphism $g: A \rightarrow B$ in \mathcal{C} with a section of the \mathcal{V} -natural transformation $\varphi_j: \mathcal{C}(\text{cod } Uj, A) \rightarrow \mathcal{C}^2(Uj, g)$ given by $d \mapsto g \cdot d \cdot Uj$. A morphism $(g, \varphi) \rightarrow (g', \varphi')$ is a morphism $(h, k): g \rightarrow g'$ in \mathcal{C}^2 such that $h \cdot \varphi_j(u, v) = \varphi'_j(h \cdot u, k \cdot v)$; 2-cells are similarly described. Objects of \mathbb{J}^\flat , first described in [4], are objects (g, φ) of \mathcal{J}^\flat that satisfy: for any composable pair (j, i) of objects of \mathcal{J} ,

$$\varphi_{j \bullet i}(h, k) = \varphi_j(\varphi_i(h, k \cdot Uj), k).$$

Lemma 6. *Let (\mathcal{I}, U) and (\mathcal{J}, V) be two \mathcal{V} -categories over \mathcal{C}^2 . Then there is an isomorphism between $(\mathcal{I} + \mathcal{J}, \binom{U}{V})^{\flat_M}$ and the pullback of $\mathcal{I}^{\flat_M} \times \mathcal{J}^{\flat_M} \rightarrow \mathcal{C}^2 \times \mathcal{C}^2$ along the diagonal $\mathcal{C}^2 \rightarrow \mathcal{C}^2 \times \mathcal{C}^2$.*

Proof. The \mathcal{V} -functor $W_{\binom{U}{V}}: \mathcal{C}^2 \rightarrow [\mathcal{I}^{\text{op}} + \mathcal{J}^{\text{op}}, \mathcal{V}]^2$ can easily be seen to be

$$\mathcal{C}^2 \xrightarrow{\Delta} \mathcal{C}^2 \times \mathcal{C}^2 \xrightarrow{W_U \times W_V} [\mathcal{I}^{\text{op}}, \mathcal{V}]^2 \times [\mathcal{J}^{\text{op}}, \mathcal{V}]^2 \cong [\mathcal{I}^{\text{op}} + \mathcal{J}^{\text{op}}, \mathcal{V}]^2. \quad (7)$$

Since the \mathcal{V} -monad M on $[\mathcal{I}^{\text{op}} + \mathcal{J}^{\text{op}}, \mathcal{V}]^2$ is obtained by pointwise application of the \mathcal{V} -monad of the same name on \mathcal{V}^2 , the isomorphism in (7) induces an isomorphism

$$\begin{array}{ccc} M\text{-Alg} \times M\text{-Alg} & \xrightarrow{\cong} & M\text{-Alg} \\ \downarrow & & \downarrow \\ [\mathcal{I}^{\text{op}}, \mathcal{V}]^2 \times [\mathcal{J}^{\text{op}}, \mathcal{V}]^2 & \xrightarrow{\cong} & [\mathcal{I}^{\text{op}} + \mathcal{J}^{\text{op}}, \mathcal{V}]^2 \end{array}$$

The rest of the proof is straightforward. \square

Lemma 7. *Suppose that $U: \mathcal{J} \rightarrow \mathcal{C}^2$ factors through the identities \mathcal{V} -functor $\text{id}: \mathcal{C} \rightarrow \mathcal{C}^2$. Then $\mathcal{J}^{\flat_M} \cong \mathcal{C}^2$ over \mathcal{C}^2 .*

Proof. Suppose that $U = \text{id } V$ for a \mathcal{V} -functor $V: \mathcal{J} \rightarrow \mathcal{C}$. Note that $\tilde{U}(g) = \mathcal{C}^2(\text{id } V -, g) \cong \mathcal{C}(V -, \text{dom } g)$, so $\tilde{U} \cong \tilde{V} \text{ dom} = \tilde{V} \mathcal{C}^{\delta_1}$, where $\delta_1: \mathbf{1} \rightarrow \mathbf{2}$ is the functor that misses out the terminal object $1 \in \mathbf{2}$. Therefore,

$$W_U = \tilde{U}^2 \mathcal{C}^t \cong \tilde{V}^2 (\mathcal{C}^{\delta_1})^2 \mathcal{C}^t \cong \tilde{V}^2 \mathcal{C}^{\delta_1 \times 2} \mathcal{C}^t = \tilde{V}^2 \mathcal{C}^{t \cdot (\delta_1 \times 2)} = \tilde{V}^2 \text{id dom} = \text{id } \tilde{V} \text{ dom}$$

where we have identified $(\mathcal{C}^2)^2$ with $\mathcal{C}^{2 \times 2}$ and used that $t \cdot (\delta_1 \times \mathbf{2}): \mathbf{2} \rightarrow \mathbf{2} \times \mathbf{2} \rightarrow \mathbf{2}$ is the functor $\mathbf{2} \rightarrow \mathbf{1} \rightarrow \mathbf{2}$ constant at $0 \in \mathbf{2}$ – see Remark 2 for the definition of t . The proof is concluded by noting that identity morphisms have a unique M -algebra structure, since M is codomain-preserving. Thus, $W_U(f)$ has a unique M -algebra structure, or in other words, $\mathcal{J}^{\flat_M} \rightarrow \mathcal{C}^2$ is bijective on objects. The rest of the proof is similarly deduced from this argument. \square

Lemma 8. *Each internal category over $\text{Sq}(\mathcal{C})$ of the form \mathcal{J}^{\flat_M} can be realised as \mathbb{I}^{\flat_M} for an internal category \mathbb{I} .*

Proof. Given a \mathcal{V} -category \mathcal{J} , consider the internal graph given by the coproduct coprojections $(\iota_0, \iota_1): \mathcal{J} \rightrightarrows \mathcal{J} + \mathcal{J} = 2 \cdot \mathcal{J}$. We claim that the free internal category on this graph exists. To prove this, we first make a digression.

An internal graph with object of objects X_0 in a finitely complete category \mathcal{W} can be seen as a span, an object of the category $\mathbf{Spn}(\mathcal{W})(X_0, X_0) = \mathcal{W}/X_0 \times X_0$. This category is monoidal, with tensor product given by composition of spans: the composition of $(a_0, a_1): A \rightrightarrows X_0$ with $(b_0, b_1): B \rightrightarrows X_0$ is the pullback $A_{a_1 \times b_0} B$. An internal category with object of objects X_0 is the same as a monoid in this monoidal category, and the free internal category on an internal graph can be constructed as the free monoid on the corresponding object of $\mathbf{Spn}(\mathcal{W})(X_0, X_0)$.

Returning to the main thread of the proof, observe that, because coproducts in $\mathcal{V}\text{-Cat}$ are disjoint unions, $\mathcal{V}\text{-Cat}$ has disjoint coproducts stable under pullback. Then, $\mathbf{Spn}(\mathcal{V}\text{-Cat})(2 \cdot \mathcal{J}, 2 \cdot \mathcal{J})$ has coproducts preserved by tensor product on each

side, and by a classical result, it admits free monoids. Furthermore, free monoids are given by the classical power series formula.

The span $(\iota_0, \iota_1): \mathcal{J} \rightrightarrows \mathcal{J} + \mathcal{J}$ satisfies $\mathcal{J} \otimes \mathcal{J} \cong 0$, since coproducts are disjoint – here \otimes denotes the tensor product given by composition of spans. Thus, the free monoid $\mathbb{I} = (\mathcal{I} \rightrightarrows 2 \cdot \mathcal{J})$ on \mathcal{J} is

$$\mathcal{I} \cong \sum_{n=0}^{\infty} \mathcal{J}^{\otimes n} \cong (\mathcal{J} + \mathcal{J}) + \mathcal{J}.$$

Let $U: \mathcal{J} \rightarrow \mathcal{C}^2$ be a \mathcal{V} -functor. The map of graphs depicted below

$$\begin{array}{ccc} \mathcal{J} & \xrightarrow{U} & \mathcal{C}^2 \\ \downarrow \iota_0 & & \downarrow \text{cod} \\ \mathcal{J} + \mathcal{J} & \xrightarrow{\left(\begin{smallmatrix} \text{dom } U \\ \text{cod } U \end{smallmatrix} \right)} & \mathcal{C} \end{array}$$

induces an internal functor $\mathbb{I} \rightarrow \text{Sq}(\mathcal{C})$, given on objects of objects by $\left(\begin{smallmatrix} \text{dom } U \\ \text{cod } U \end{smallmatrix} \right)$ and on objects of morphism by the \mathcal{V} -functor $(\mathcal{J} + \mathcal{J}) + \mathcal{J} \rightarrow \mathcal{C}^2$ given on the first summand by $\text{id}\left(\begin{smallmatrix} \text{dom } U \\ \text{cod } U \end{smallmatrix}\right): \mathcal{J} + \mathcal{J} \rightarrow \mathcal{C} \rightarrow \mathcal{C}^2$ and on the second summand by $U: \mathcal{J} \rightarrow \mathcal{C}^2$.

Lemma 6 allows us to calculate \mathbb{I}^{\clubsuit_M} . First, $\mathcal{I} \otimes \mathcal{I} = \mathcal{I} \times_{2 \cdot \mathcal{J}} \mathcal{I}$ is

$$(2 \cdot \mathcal{J}) + \mathcal{J} + \mathcal{J} \rightrightarrows 2 \cdot \mathcal{J}$$

and $(\mathcal{I} \times_{2 \cdot \mathcal{J}} \mathcal{I})^{\clubsuit_M}$ is the pullback of

$$(2 \cdot \mathcal{J})^{\clubsuit_M} \times \mathcal{J}^{\clubsuit_M} \times \mathcal{J}^{\clubsuit_M} \longrightarrow \mathcal{C}^2 \times \mathcal{C}^2 \times \mathcal{C}^2 \quad (8)$$

along the diagonal $\mathcal{C}^2 \rightarrow \mathcal{C}^2 \times \mathcal{C}^2 \times \mathcal{C}^2$. One can easily compute each factor in (8). Recall that the \mathcal{V} -functor $2 \cdot \mathcal{J} \rightarrow \mathcal{C}^2$ is $\text{id}\left(\begin{smallmatrix} \text{dom } U \\ \text{cod } U \end{smallmatrix}\right)$, so $(2 \cdot \mathcal{J})^{\clubsuit_M}$ is isomorphic to \mathcal{C}^2 over \mathcal{C}^2 by Lemma 7. Therefore, $(\mathcal{I} \times_{2 \cdot \mathcal{J}} \mathcal{I})^{\clubsuit_M}$ is the pullback of

$$\mathcal{C}^2 \times \mathcal{J}^{\clubsuit_M} \times \mathcal{J}^{\clubsuit_M} \xrightarrow{1 \times U^{\clubsuit_M} \times U^{\clubsuit_M}} \mathcal{C}^2 \times \mathcal{C}^2 \times \mathcal{C}^2$$

along the diagonal, which is clearly isomorphic to $U^{\clubsuit_M}: \mathcal{J}^{\clubsuit_M} \rightarrow \mathcal{C}^2$.

By the argument given in the previous paragraph, $\mathcal{I}^{\clubsuit_M} \cong \mathcal{J}^{\clubsuit_M}$, and the parallel \mathcal{V} -functors $\mathcal{I}^{\clubsuit_M} \rightrightarrows (\mathcal{I} \times_{2 \cdot \mathcal{J}} \mathcal{I})^{\clubsuit_M}$ of (6) both become, under the isomorphisms we have exhibited, the identity functor of $\mathcal{J}^{\clubsuit_M}$. Their equaliser \mathbb{I}^{\clubsuit_M} is, therefore, isomorphic to $\mathcal{J}^{\clubsuit_M}$, completing the proof. \square

4. LAX ORTHOGONAL FACTORISATION SYSTEMS

This section is a short exposition of the basic definitions of lax orthogonal AWFSS [6], beginning with lax idempotent 2-monads – which are closely related to Kock–Zöberlein doctrines [18, 23].

Recall that a 2-monad $\mathbb{T} = (T, i, m)$ is *lax idempotent* if one of the following equivalent conditions holds: (1) $Ti \dashv m$ with identity unit; (2) $m \dashv iT$ with identity counit; (3) there exists a modification $\delta: Ti \Rrightarrow iT$ such that $m \cdot \delta = 1$ and $\delta \cdot i = 1$; (4) the forgetful 2-functor from the 2-category of \mathbb{T} -algebras and lax morphisms is full and faithful on morphisms. Some of these conditions appear in [18] and [23] in the context of doctrines. The equivalence between the first two conditions in the list first appeared in [20], and a full list of equivalent conditions with the respective proofs can be found in [17].

A 2-comonad \mathbb{G} on \mathcal{K} is lax idempotent when the corresponding monad \mathbb{G}^{op} on the 2-category \mathcal{K}^{op} obtained by reversing the morphisms is lax idempotent.

Definition 9 ([6]). A \mathcal{V} -enriched AWFSS (\mathbb{L}, \mathbb{R}) on \mathcal{C} is *lax orthogonal* if the 2-comonad \mathbb{L} and the 2-monad \mathbb{R} are lax idempotent, or equivalently, as shown in [6, §4], if either \mathbb{L} or \mathbb{R} is lax idempotent.

Example 10. The two AWFS of Example 2.4 are lax orthogonal. It is easy to verify that the \mathcal{V} -monad \mathbf{R} – the free split opfibration monad – is lax idempotent, and by a duality argument, so it is the comonad \mathbf{E} .

Assumption 11. For the rest of the section we equip \mathcal{V} with the lax orthogonal AWFS (\mathbf{E}, \mathbf{M}) described in Example 2.4 – so \mathbf{M} is the free RALI \mathcal{V} -monad.

Lax orthogonality is closely related to the notion of KZ-lifting operation [6, §5]. If f and g are morphisms in \mathcal{C} , a KZ-lifting operation from f to g is a RALI structure on the dashed morphism induced by the universal property of pullbacks. If a KZ-lifting operation from f to g exists we say that f and g are *KZ-orthogonal*.

$$\begin{array}{ccccc}
 \mathcal{C}(\text{cod } f, \text{dom } g) & & \xrightarrow{\mathcal{C}(f,1)} & & \mathcal{C}(\text{dom } f, \text{dom } g) \\
 & \searrow \text{dashed} & & \searrow & \\
 & \mathcal{C}^2(f, g) & \xrightarrow{\quad} & & \mathcal{C}(\text{dom } f, \text{dom } g) \\
 & \downarrow \mathcal{C}(1,g) & & \downarrow \mathcal{C}(1,g) & \\
 \mathcal{C}(\text{cod } f, \text{cod } g) & \xrightarrow{\mathcal{C}(f,1)} & & \mathcal{C}(\text{dom } f, \text{cod } g) &
 \end{array}$$

Slightly more generally, a KZ-lifting operation from a \mathcal{V} -functor $U: \mathcal{A} \rightarrow \mathcal{C}^2$ to another $V: \mathcal{B} \rightarrow \mathcal{C}^2$ is a RALI structure on the comparison \mathcal{V} -natural transformation $\mathcal{C}(\text{cod } U, \text{dom } V) \Rightarrow \mathcal{C}^2(U, V)$. Clearly, if a KZ-lifting operations are unique up to isomorphism.

Furthermore, given a KZ-lifting operation as the one in the previous paragraph and \mathcal{V} -functors $U': \mathcal{A} \rightarrow \mathcal{A}$ and $V': \mathcal{B}' \rightarrow \mathcal{B}$, there is an “restricted” KZ-lifting operation from UU' to VV' .

Notation 12. When \mathcal{V} is equipped with the AWFS (\mathbf{E}, \mathbf{M}) whose \mathbf{M} -algebras are the RALIs in \mathcal{V} – see Example 2.4 – then $\mathbb{J}^{\mathfrak{h}\mathbf{M}}$ and $\mathcal{J}^{\mathfrak{h}\mathbf{M}}$ will be denoted by $\mathbb{J}^{\mathfrak{h}\mathbf{KZ}}$ and $\mathcal{J}^{\mathfrak{h}\mathbf{KZ}}$. The latter is denoted in [6] by $\mathcal{J}^{\mathfrak{p}}$, and fits, by definition, the following pullback square.

$$\begin{array}{ccc}
 \mathcal{J}^{\mathfrak{h}\mathbf{KZ}} & \longrightarrow & \mathbf{Rali}[\mathcal{J}^{\text{op}}, \mathcal{V}] \\
 \downarrow U^{\mathfrak{h}\mathbf{KZ}} & & \downarrow \\
 \mathcal{C}^2 & \xrightarrow{W_U} & [\mathcal{J}^{\text{op}}, \mathcal{V}]^2
 \end{array}$$

Remark 13. If $U: \mathcal{J} \rightarrow \mathcal{C}^2$ is a small \mathcal{V} -category over \mathcal{C}^2 , then $U^{\mathfrak{h}\mathbf{KZ}}: \mathcal{J}^{\mathfrak{h}\mathbf{KZ}} \rightarrow \mathcal{C}^2$ is the universal \mathcal{V} -category over \mathcal{C}^2 equipped with a KZ-lifting operation from U to $U^{\mathfrak{h}\mathbf{KZ}}$ – see [6, §6].

Theorem 14 ([6, Thm. 6.6]). *A \mathcal{V} -enriched AWFS (\mathbf{L}, \mathbf{R}) on \mathcal{C} is lax orthogonal if and only if there exists a KZ-lifting operation from the forgetful $U: \mathbf{L}\text{-Coalg} \rightarrow \mathcal{C}^2$ to the forgetful $V: \mathbf{R}\text{-Alg} \rightarrow \mathcal{C}^2$.*

In particular, any \mathbf{L} -coalgebra is KZ-orthogonal to any \mathbf{R} -algebra.

Theorem 15. *A \mathcal{V} -enriched AWFS (\mathbf{L}, \mathbf{R}) on a \mathcal{V} -category \mathcal{C} is lax orthogonal if and only if $\mathbf{R}\text{-Alg} \cong \mathbf{L}\text{-Coalg}^{\mathfrak{h}\mathbf{KZ}}$ over \mathcal{C}^2 .*

Proof. We use Notation 5. First observe that, for any \mathcal{V} -category \mathcal{A} over \mathcal{C}^2 the displayed square is a pullback, because RALIs are split epis. Moreover, the horizontal \mathcal{V} -functors are full and faithful.

$$\begin{array}{ccc}
 \mathcal{A}^{\mathfrak{h}\mathbf{KZ}} & \longrightarrow & \mathcal{A}^{\mathfrak{h}\mathbf{KZ}} \\
 \downarrow & & \downarrow \\
 \mathcal{A}^{\mathfrak{h}} & \longrightarrow & \mathcal{A}^{\mathfrak{h}}
 \end{array}$$

Theorem 6.6 of [6] shows that there is an – essentially unique – \mathcal{V} -functor $\mathbf{R}\text{-Alg} \rightarrow \mathbf{L}\text{-Coalg}^{\mathfrak{h}_{\text{KZ}}}$, commuting with the forgetful functors into \mathcal{C}^2 , which is, moreover, full and faithful. On the other hand, there is an isomorphism $\mathbf{R}\text{-Alg} \rightarrow \mathbf{L}\text{-Coalg}^{\mathfrak{h}}$ over \mathcal{C}^2 . We can deduce the existence of the required isomorphism as in the statement from the pullback of the previous paragraph. \square

Definition 16. We say that a \mathcal{V} -enriched AWFS (\mathbf{L}, \mathbf{R}) is *cofibrantly KZ-generated* by a small internal category \mathbb{J} in $\mathcal{V}\text{-Cat}$ over $\mathbb{S}\mathbf{q}(\mathcal{C})$ if $\mathbf{R}\text{-Alg} \cong \mathbb{J}^{\mathfrak{h}_{\text{KZ}}}$ over $\mathbb{S}\mathbf{q}(\mathcal{C})$. A \mathcal{V} -enriched AWFS is cofibrantly KZ-generated if there exists such a small internal category in $\mathcal{V}\text{-Cat}$.

5. LAX ORTHOGONALITY OF KZ-COFIBRANTLY GENERATED AWFSs

If (\mathbf{L}, \mathbf{R}) is cofibrantly KZ-generated by \mathbb{J} , it is clear that each Uj is KZ-orthogonal to each \mathbf{R} -algebra – Remark 13. It is not completely obvious, however, that each \mathbf{L} -coalgebra is KZ-orthogonal to each \mathbf{R} -algebra, and the proof of this fact occupies the remaining of the section.

Now assume that \mathcal{C} is a – not necessarily small – complete \mathcal{V} -category; strictly speaking, all we need is cotensor products and equalisers. For each pair of objects A and B denote by $\langle A, B \rangle$ the right Kan extension of the \mathcal{V} -functor $A: \mathbf{1} \rightarrow \mathcal{C}$ along $B: \mathbf{1} \rightarrow \mathcal{C}$.

$$\langle A, B \rangle = \text{Ran}_A B = \{\mathcal{C}(-, A), B\}$$

This defines a **CAT**-functor $\langle -, - \rangle: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{End}(\mathcal{C})$. As for any right Kan extension of a functor along itself, $\langle A, A \rangle = \text{Ran}_A A$ has a canonical structure of a \mathcal{V} -monad.

For a morphism $f: A \rightarrow B$ in \mathcal{C} , there is a \mathcal{V} -monad $\langle f, f \rangle_\ell$ defined as the comma-object

$$\begin{array}{ccccc} & & \langle A, A \rangle & \xrightarrow{\langle A, f \rangle} & \langle A, B \rangle \\ & \nearrow & \Downarrow & \searrow & \\ \langle f, f \rangle_\ell & & & & \\ & \searrow & \langle B, B \rangle & \xleftarrow{\langle f, B \rangle} & \end{array} \quad (9)$$

The monad structure on $\langle f, f \rangle_\ell$ is such that the projections of the comma-object are – strict – morphisms of \mathcal{V} -monads. This method of describing morphisms as monad maps can be found as early as [14, §3].

We shall denote the category of \mathcal{V} -monads on \mathcal{C} and – strict – monad morphisms – monoid morphisms in the monoidal category of endo- \mathcal{V} -functors – by $\mathbf{Mnd}(\mathcal{C})$.

The definition of $\langle A, A \rangle$ is such that there is a bijection between morphisms $\mathbf{T} \rightarrow \langle A, A \rangle$ in $\mathbf{Mnd}(\mathcal{C})$ and \mathbf{T} -algebra structures on A . Similarly, there is a bijection between morphisms $\mathbf{T} \rightarrow \langle f, f \rangle_\ell$ in $\mathbf{Mnd}(\mathcal{C})$ and: \mathbf{T} -algebra structures on A and on B together with a lax morphism structure on f .

Remark 17. Following [17], if we denote by $\sigma: \langle f, f \rangle_\ell \rightarrow \langle A, A \rangle \times \langle B, B \rangle$ the morphism of \mathcal{V} -monads induced by the projections of the comma-object (9), then \mathbf{T} is lax idempotent precisely when it is co-orthogonal to σ in $\mathbf{Mnd}(\mathcal{C})$; ie when $\mathbf{Mnd}(\mathcal{C})(\mathbf{T}, \sigma)$ is a bijection.

Lemma 18. Suppose that $W: \mathcal{C} \rightarrow \mathcal{B}$ is a continuous \mathcal{V} -functor between complete \mathcal{V} -categories. Then, the canonical \mathcal{V} -natural transformation

$$\text{Ran}_W(W\langle A, B \rangle) \longrightarrow \text{Ran}_{W(A)}(W(B))$$

is an isomorphism. When $A = B$, it is an isomorphism of \mathcal{V} -monads. If f, g are morphisms in \mathcal{C} , then the canonical \mathcal{V} -natural transformation

$$\text{Ran}_W(W\langle f, g \rangle_\ell) \longrightarrow \langle Wf, Wg \rangle_\ell$$

is an isomorphism. When $f = g$ it is an isomorphism of \mathcal{V} -monads.

Proof. The continuous \mathcal{V} -functor W preserves the limits $\langle A, B \rangle(X) = \{\mathcal{C}(X, A), B\}$, so $W\langle A, B \rangle \cong \text{Ran}_A(W(B))$. Thus, $\text{Ran}_W(W\langle A, B \rangle) \cong \text{Ran}_W \text{Ran}_A(W(B)) \cong \text{Ran}_{W(A)}(W(B))$.

To prove the second part of the statement, note that W preserves the comma object (9), as it is defined by componentwise comma objects. Now use the isomorphisms of the first part to show that $\text{Ran}_W(W\langle f, g \rangle_\ell)$ is the comma object of the cospan $\langle Wf, WB \rangle, \langle WA, Wg \rangle$. \square

If $W: \mathcal{C} \rightarrow \mathcal{B}$ is a \mathcal{V} -functor and $\mathsf{T} = (T, i, m)$ a \mathcal{V} -monad on \mathcal{C} , then $\text{Ran}_W(WT)$ has a canonical structure of a \mathcal{V} -monad on \mathcal{B} , whenever this right Kan extension exists. The unit is the \mathcal{V} -natural transformation that corresponds to $Wi: W \Rightarrow WT$, while the multiplication corresponds to the \mathcal{V} -natural transformation

$$\text{Ran}_W(WT) \text{Ran}_W(WT)W \longrightarrow \text{Ran}_W(WT)WT \longrightarrow WTT \xrightarrow{Wm} WT.$$

If Ran_W always exists, then $E \mapsto \text{Ran}_W(WE)$ is a monoidal functor from $\text{End}(\mathcal{C})$ to $\text{End}(\mathcal{B})$, so it induces a functor

$$\mathbf{Mnd}(\mathcal{C}) \longrightarrow \mathbf{Mnd}(\mathcal{B}). \quad (10)$$

This is the case, for example, when W has a left adjoint.

In a moment we will need a more general notion of morphism of \mathcal{V} -monads. If $(\mathcal{C}, \mathsf{T})$ is a \mathcal{V} -monad on \mathcal{C} and $(\mathcal{B}, \mathsf{S})$ a \mathcal{V} -monad on \mathcal{B} , a morphism $(\mathcal{C}, \mathsf{T}) \rightarrow (\mathcal{B}, \mathsf{S})$ is a \mathcal{V} -functor $F: \mathcal{C} \rightarrow \mathcal{B}$ equipped with a \mathcal{V} -natural transformation $\varphi: SF \Rightarrow FT$ satisfying compatibility axioms with the unit and multiplication of the respective monads. The effect of these axioms is that, if $a: TA \rightarrow A$ is a T -algebra, then $Fa \cdot \varphi_A: SFA \rightarrow FTA \rightarrow FA$ is an S -algebra.

There is a bijection between morphisms of \mathcal{V} -monads $(\mathcal{C}, \mathsf{T}) \rightarrow (\mathcal{B}, \mathsf{S})$ over $W: \mathcal{C} \rightarrow \mathcal{B}$ and morphisms $\mathsf{S} \rightarrow \text{Ran}_W(WT)$ in $\mathbf{Mnd}(\mathcal{C})$. Indeed, a morphism structure $\omega: SW \Rightarrow WT$ on W corresponds to a morphism $\mathsf{S} \rightarrow \text{Ran}_W(WT)$ via the universal property of Ran_W .

Lemma 19. *Consider a commutative diagram of \mathcal{V} -functors as depicted, where T and S are \mathcal{V} -monads on \mathcal{C} and \mathcal{B} . Let (W, w) be the corresponding morphism of \mathcal{V} -monads, and assume that Ran_W of \mathcal{V} -functors into \mathcal{B} always exist. If the square is a pullback, then (W, w) exhibits T as a reflection of S along (10).*

$$\begin{array}{ccc} \mathsf{T}\text{-Alg} & \longrightarrow & \mathsf{S}\text{-Alg} \\ \downarrow V^{\mathsf{T}} & & \downarrow V^{\mathsf{S}} \\ \mathcal{C} & \xrightarrow{W} & \mathcal{B} \end{array} \quad (11)$$

Proof. Let P be a \mathcal{V} -monad on \mathcal{C} and consider the following sets: (1) $\mathbf{Mnd}(\mathcal{C})(\mathsf{T}, \mathsf{P})$; (2) \mathcal{V} -functors $\mathsf{P}\text{-Alg}_s \rightarrow \mathsf{T}\text{-Alg}_s$ that commute with the respective forgetful functors; (3) $\mathbf{Mnd}(\mathcal{B})(\mathsf{S}, \text{Ran}_W(WP))$; (4) \mathcal{V} -functors $\mathsf{P}\text{-Alg} \rightarrow \mathsf{S}\text{-Alg}$ over W . The sets (1) and (2) and the sets (3) and (4) are bijective, naturally in P , by the comments previous to this lemma. To say that T is the reflection of S is equally saying that (1) is naturally bijective to (3), while to say that (11) is a pullback implies that (2) and (4) are naturally bijective. \square

The hypothesis of the existence of Ran_W in Lemma 19 is satisfied in two common situations: when W has a left adjoint; and when \mathcal{C} has a small codense category and \mathcal{B} is complete.

Corollary 20. *If the \mathcal{V} -monad S in Lemma 19 is lax idempotent, both \mathcal{C} , \mathcal{B} are complete and W is continuous, then T is lax idempotent.*

Proof. By Remark 17, we must show that \mathbf{T} is co-orthogonal to the morphism $\sigma_f: \langle f, f \rangle_\ell \rightarrow \langle A, A \rangle \times \langle B, B \rangle$, for any morphism $f: A \rightarrow B$ in \mathcal{C} . Equivalently, that \mathbf{S} is co-orthogonal to $\text{Ran}_W(W\sigma)$ by Lemma 19. This morphism is isomorphic to σ_{Wf} , via the isomorphisms $\text{Ran}_W(W\langle A, A \rangle) \cong \langle (W(A), W(A)) \rangle$ and $\text{Ran}_W(W\langle f, f \rangle_\ell) \cong \langle W(f), W(f) \rangle_\ell$ of Lemma 18, from where it is obvious that \mathbf{S} is co-orthogonal to $\text{Ran}_W(W\sigma)$. \square

Theorem 21. *On any complete and cocomplete \mathcal{V} -category, any \mathcal{V} -enriched AWFSs that is cofibrantly KZ-generated by a small \mathcal{V} -category is lax orthogonal.*

Proof. Suppose that the \mathcal{V} -enriched AWFS (\mathbf{L}, \mathbf{R}) on \mathcal{C} is cofibrantly KZ-generated by a \mathcal{V} -category $\mathcal{J} \rightarrow \mathcal{C}^2$ over \mathcal{C}^2 . Then we have that the forgetful \mathcal{V} -functor $\mathbf{R}\text{-Alg} \rightarrow \mathcal{C}^2$ is the pullback of $\mathbf{M}\text{-Alg} \rightarrow [\mathcal{J}^{\text{op}}, \mathcal{V}]^2$ along W_U . Thus, Corollary 20 will ensure that \mathbf{R} is lax idempotent once we have shown that the right Kan extensions Ran_{W_U} exist. The latter condition is obvious since W_U has a left adjoint – see Remark 2. \square

Theorem 22. *Suppose that \mathcal{C} is a \mathcal{V} -category with the property that any small \mathcal{V} -category over \mathcal{C}^2 cofibrantly KZ-generates an awfs. Then, any AWFS cofibrantly KZ-generated by a small internal category in $\mathcal{V}\text{-Cat}$ is lax orthogonal.*

Proof. Suppose that \mathbb{J} cofibrantly KZ-generates the AWFS (\mathbf{L}, \mathbf{R}) . By considering the equaliser (6), we can exhibit \mathbb{J}^{thm} as the equaliser of a pair of \mathcal{V} -functors, as displayed in the left below, where \mathbf{T} and \mathbf{S} are the monad part of the lax orthogonal AWFSs generated by \mathcal{J}^{thm} and $(\mathcal{J} \times_{\mathcal{J}_0} \mathcal{J})^{\text{thm}}$, respectively. In fact, in what follows we only need regard \mathbf{R} , \mathbf{T} and \mathbf{S} as ordinary monads on the ordinary category \mathcal{C}_\circ^2 . Since any functor between categories of algebras that commutes with the respective forgetful functors is induced by a unique morphism of monads, there is a commutative diagram in the category $\mathbf{Mnd}(\mathcal{C}_\circ^2)$ as displayed on the right hand side.

$$\mathbf{R}\text{-Alg} \cong \mathbb{J}^{\text{thm}} \longrightarrow \mathbf{T}\text{-Alg} \rightrightarrows \mathbf{S}\text{-Alg} \quad \mathbf{S}_\circ \begin{array}{c} \xrightarrow{\tau} \\ \xleftarrow{\nu} \end{array} \mathbf{T}_\circ \xrightarrow{\zeta} \mathbf{R}_\circ$$

By definition, this diagram induces an equaliser diagram at the level of corresponding categories of algebras over \mathcal{C}_\circ^2 , so it exhibits the ordinary monad \mathbf{R}_\circ as the algebraic coequaliser of τ and ν , and by [15, Prop. 26.2], as the coequaliser of this pair of morphisms in the category $\mathbf{Mnd}(\mathcal{C}_\circ^2)$ of – ordinary – monads on \mathcal{C}_\circ^2 .

We have to prove that \mathbf{R}_\circ is co-orthogonal to the morphism $\sigma_f: \langle f, f \rangle_\ell \rightarrow \langle A, A \rangle \times \langle B, B \rangle$, for $f: A \rightarrow B$ in \mathcal{C} – Remark 17. To give a morphism $\mathbf{R}_\circ \rightarrow \text{cod}(\sigma_f)$ is equally well to give a morphism $\alpha: \mathbf{T}_\circ \rightarrow \text{cod}(\sigma_f)$ such that $\alpha \cdot \tau = \alpha \cdot \nu$. Since \mathbf{T} is lax idempotent, there exists a unique $\beta: \mathbf{T}_\circ \rightarrow \text{dom}(\sigma_f)$ such that $\sigma \cdot \beta = \alpha$, again by Remark 17. The equality $\sigma \cdot \beta \cdot \tau = \sigma \cdot \beta \cdot \nu$ is easily derived, hence $\beta \cdot \tau = \beta \cdot \nu$ since \mathbf{S} is lax idempotent, and $\beta = \gamma \cdot \zeta$ for a unique $\gamma: \mathbf{R}_\circ \rightarrow \text{dom}(\sigma)_\circ$. Therefore $\sigma \cdot \gamma$ is the original morphism $\mathbf{R}_\circ \rightarrow \text{cod}(\sigma)_\circ$, and we have established the desired bijection. \square

In certain conditions, the hypothesis of Theorem 22 are always satisfied, as for example:

Corollary 23. *Let \mathcal{C} be a locally presentable \mathcal{V} -category. Then, any small internal category \mathbb{J} in $\mathcal{V}\text{-Cat}$ over $\text{Sq}(\mathcal{C})$ cofibrantly KZ-generates a \mathcal{V} -enriched AWFS which, moreover, is lax orthogonal.*

In the corollary, the existence of the AWFS cofibrantly KZ-generated by \mathbb{J} can be deduced from [4, §8.2], which shows the existence of a **Set**-enriched AWFS. The extension to include the 2-cells can be made without much effort, and is left to the reader, obtaining in this way a \mathcal{V} -enriched AWFS, which is necessarily lax orthogonal by Theorem 22.

6. SOME CONSEQUENCES OF COFIBRANT GENERATION

In this section we show that cofibrantly generated AWFSS and WFSs on categories that satisfy a smallness condition with respect to an orthogonal factorisation system – abbreviated OFS.

An OFS can be succinctly described as a WFS whose diagonal fillers are unique – even though OFS appeared independently in [10]. Recall that an OFS $(\mathcal{E}, \mathcal{M})$ on a cocomplete category is *cocomplete* if pushouts of arbitrary families of morphisms in \mathcal{E} with the same domain exist; in any such OFS, the morphisms in \mathcal{E} must be epimorphisms – see [15, §1.3]. The category is *cowellpowered* if there is, up to isomorphism, only a small set of morphisms in \mathcal{E} with a given domain. A morphism in \mathcal{M} with codomain B is called an \mathcal{M} -*subobject*.

Let \mathcal{A} be a cocomplete category with a cocomplete OFS $(\mathcal{E}, \mathcal{M})$. Recall that an object A of \mathcal{A} is said to have *rank* less or equal to a regular cardinal κ if for any κ -filtered co-cone $m_\beta: X_\beta \rightarrow X$, with $m_\beta \in \mathcal{M}$, the functor $\mathcal{A}(A, -)$ preserves $\text{colim } X_\beta$. Observe that each morphism $X_\beta \rightarrow X_\gamma$ of the diagram is in \mathcal{M} since \mathcal{E} consists of epimorphisms – see [10, 2.1.4]. It is clear that A has rank less or equal to κ if and only if $\mathcal{A}(A, -)$ preserves the colimit of all κ -filtered diagrams in \mathcal{M} . The rank of A is the smallest regular cardinal κ for which this happens. This notion of rank is attributed by [10] to M. Barr. Observe that the induced morphism $\text{colim } X_\beta \rightarrow X$ need not be in \mathcal{M} ; indeed, it is in \mathcal{M} precisely when $\text{colim } X_\beta$ is the union of the subobjects X_β of X .

The objects of locally presentable categories have a rank for the OFSs (Iso, Mor) . Categories locally bounded with respect to a OFS – in the sense of [16, §6.1] – have ranked objects, due to the argument given in [10, §3.2]. A number examples are given in [10] and [16, §6.1]; later we will be interested in the example of the category of T_0 topological spaces \mathbf{Top}_0 equipped with the OFS $\mathcal{E} = \text{surjections}$ and $\mathcal{M} = \text{subspace inclusions}$.

Each cocomplete orthogonal factorisation system $(\mathcal{E}, \mathcal{M})$ on \mathcal{A} induces another on \mathcal{A}^2 component-wise, which we still call $(\mathcal{E}, \mathcal{M})$. It is easy to show that a morphism $f: X \rightarrow Y$ has rank less or equal to κ as an object of \mathcal{A}^2 if X and Y have rank less or equal to κ .

An AWFSS (\mathbf{E}, \mathbf{M}) on \mathcal{C} is *cocontinuous* if one of the following equivalent conditions hold: the comonad \mathbf{E} is cocontinuous; the monad \mathbf{M} is cocontinuous; the \mathcal{V} -functor $K: \mathcal{C}^2 \rightarrow \mathcal{C}$, part of the \mathcal{V} -functorial factorisation, is cocontinuous.

Proposition 24. *Suppose \mathcal{V} is equipped with a cocontinuous AWFSS (\mathbf{E}, \mathbf{M}) and \mathcal{C} a cocomplete \mathcal{V} -category whose underlying category is locally ranked with respect to some proper OFS. If \mathbb{J} is a small internal category in $\mathcal{V}\text{-Cat}$ over $\text{Sq}(\mathcal{C})$, then $\mathbb{J}^{\mathbf{M}} \rightarrow \mathcal{C}^2$ creates κ -filtered colimits of \mathcal{M} -subobjects.*

Proof. We first prove the case of cofibrant generation by a small category \mathcal{J} over \mathcal{C}^2 . By definition – see diagram (2) – $\mathcal{J}^{\mathbf{M}} \rightarrow \mathcal{C}^2$ is the pullback of the forgetful $V: \mathbf{M}\text{-Alg} \rightarrow [\mathcal{J}^{\text{op}}, \mathcal{V}]^2$ along W_U . A diagram $D: \mathcal{D} \rightarrow \mathcal{J}^{\mathbf{M}}$ is equivalently given by a pair of diagrams $D_0: \mathcal{D} \rightarrow \mathbf{M}\text{-Alg}$ and $D_1: \mathcal{D} \rightarrow \mathcal{C}^2$ satisfying $W_U D_1 = V D_0$. The forgetful functor $U^{\mathbf{M}}$ creates the colimit of D if W_U preserves the colimit of D_1 and the resulting colimit in $[\mathcal{J}^{\text{op}}, \mathcal{V}]^2$ is created by V .

In the case at hand, the enriched monad \mathbf{M} is cocontinuous, so V creates all colimits. It remains to verify that W_U preserves κ -filtered colimits of \mathcal{M} -subobjects. Since \mathcal{J} is small, there is a regular cardinal κ such that each $Uj \in \mathcal{C}^2$ has rank less or equal to κ , ie $\mathcal{C}^2(Uj, -)$ preserves κ -filtered colimits of \mathcal{M} -subobjects. This means that $\tilde{U}: \mathcal{C}^2 \rightarrow [\mathcal{J}^{\text{op}}, \mathcal{V}]$ preserves κ -filtered colimits of \mathcal{M} -subobjects. Furthermore, the functor W_U is the composition of \mathcal{C}^t with \tilde{U}^2 , where the former is cocontinuous – Remark 2 – and sends \mathcal{M} -subobjects of \mathcal{C}^2 to \mathcal{M} -subobjects of $\mathcal{C}^{2 \times 2} \cong (\mathcal{C}^2)^2$,

since the OFS on these categories are defined pointwise from the original OFS on \mathcal{C} . It follows that W_U preserves the required colimits.

Now we prove the case of cofibrant generation by a internal category $\mathbb{J} = (\mathcal{J} \rightrightarrows \mathcal{J}_0)$ in $\mathcal{V}\text{-Cat}$ and $U = (U_1, U_0): \mathbb{J} \rightarrow \text{Sq}(\mathcal{C})$ an internal functor. Denote by V the functor $\mathcal{J} \times_{\mathcal{J}_0} \mathcal{J} \rightarrow \mathcal{C}^2$ from the object of composable pairs. Recall from (6) that $(\mathbb{J}^{\clubsuit_M}, U^{\clubsuit_M})$ is an equaliser in $\mathcal{V}\text{-Cat}/\mathcal{C}^2$ of a pair of \mathcal{V} -functors D, E from $(\mathcal{J}_1^{\clubsuit_M}, U_1^{\clubsuit_M})$ to $((\mathcal{J}_1 \times_{\mathcal{J}_0} \mathcal{J}_1)^{\clubsuit_M}, V^{\clubsuit_M})$. Both D and E must preserve κ -filtered colimits of diagrams in $\mathcal{J}^{\clubsuit_M}$ that are sent by U^{\clubsuit_M} into \mathcal{M} , for some κ . This is because $U_1^{\clubsuit_M}$ and V^{\clubsuit_M} create, respectively, μ -filtered and ν -filtered colimits of \mathcal{M} -subobjects for regular cardinals μ, ν ; we can now take the largest of these two cardinals as κ . Since $\mathbb{J}^{\clubsuit_M} \hookrightarrow \mathcal{J}^{\clubsuit_M}$ is full and faithful, $\mathbb{J}^{\clubsuit_M} \rightarrow \mathcal{C}^2$ creates κ -filtered colimits of \mathcal{M} -subobjects, completing the proof. \square

An immediate consequence of the proposition is:

Corollary 25. *Suppose that \mathcal{V} and \mathcal{C} are as in Proposition 24. Then, any cofibrantly generated AWFS (\mathbf{L}, \mathbf{R}) on \mathcal{C} satisfies that the forgetful \mathcal{V} -functor $\mathbf{R}\text{-Alg} \rightarrow \mathcal{C}^2$ creates κ -filtered colimits of \mathcal{M} -subobjects, for some regular cardinal κ .*

Recall that the notation \mathcal{J}^{\clubsuit} , without mention of the \mathcal{V} -monad \mathbf{M} , means that we are taking as \mathbf{M} the \mathcal{V} -monad whose algebras are split epimorphisms.

Lemma 26. *Let (\mathbf{L}, \mathbf{R}) be an AWFS on the \mathcal{V} -category \mathcal{C} , with underlying WFS $(\mathcal{L}, \mathcal{R})$, and $\mathcal{I} \subset \mathcal{L}$ a small set of morphisms. There exists a \mathcal{V} -functor over \mathcal{C}^2*

$$\mathbf{R}\text{-Alg} \longrightarrow \mathcal{I}^{\clubsuit}.$$

Proof. The class \mathcal{L} consists of those morphisms of \mathcal{C} that admit at least one coalgebra structure for the copointed endo- \mathcal{V} -functor (L, Φ) . Since \mathcal{I} is small, we can choose for each $i \in \mathcal{I}$ a coalgebra structure, $(1, s_i): i \rightarrow L(i)$. We will use the explicit description of the objects of \mathcal{I}^{\clubsuit} given in [11] and before our Lemma 6. For an \mathbf{R} -algebra g , define $(g, \psi) \in \mathcal{I}^{\clubsuit}$ in the following manner. We know from [11, §3] that g has a structure of an object $(g, \varphi) \in \mathbf{L}\text{-Coalg}^{\clubsuit}$. Now define $\psi_i(h, k) = \varphi_{L(i)}(h, k \cdot \rho_i) \cdot s_i$.

This assignment on objects can be easily extended to a \mathcal{V} -functor. Note that this construction works only because \mathcal{I} is a set instead of a category, in which case we would not be able to guarantee the compatibility between the coalgebra structures chosen for each $i \in \mathcal{I}$ and the morphisms of \mathcal{I} . \square

Recall that the \mathcal{V} -monad \mathbf{R}_1 of an AWFS (\mathbf{L}, \mathbf{R}) is the fibrant replacement monad, ie the restriction of \mathbf{R} to $\mathcal{C}/1 \cong \mathcal{C}$.

Proposition 27. *Suppose that \mathcal{C} is a \mathcal{V} -category as in Proposition 24 and has a terminal object. Suppose given a \mathcal{V} -enriched AWFS (\mathbf{L}, \mathbf{R}) on \mathcal{C} . If its underlying WFS $(\mathcal{L}, \mathcal{R})$ is cofibrantly generated by a set morphisms with rank, then there is a regular cardinal κ such that any κ -filtered colimits of \mathcal{M} -subobjects $\text{colim } X_j$ is a fibrant object for $(\mathcal{L}, \mathcal{R})$, provided that X_j is a diagram in $\mathbf{R}_1\text{-Alg}$.*

Proof. Suppose that $\mathcal{I} \subset \mathcal{L}$ cofibrantly generates $(\mathcal{L}, \mathcal{R})$. Consider a \mathcal{V} -functor $\mathbf{R}\text{-Alg} \rightarrow \mathcal{I}^{\clubsuit}$ over \mathcal{C}^2 , as provided by Lemma 26. Taking fibre over the terminal object $1 \in \mathcal{C}$, we obtain the first arrow displayed below.

$$\mathbf{R}_1\text{-Alg} \longrightarrow (\mathcal{I}^{\clubsuit})_1 \longrightarrow \mathbf{Fib} \quad (12)$$

The second arrow is the inclusion into the full subcategory of fibrant objects, which exists since any object with a lifting operation against \mathcal{I} is certainly weakly orthogonal to \mathcal{I} .

Proposition 24 guarantees that the forgetful $\mathcal{I}^\heartsuit \rightarrow \mathcal{C}^2$ creates κ -filtered colimits of \mathcal{M} -subobjects for some regular cardinal κ . Therefore, if $\{X_j\}$ is a κ -filtered diagram in $\mathbf{R}_1\text{-Alg}$ of \mathcal{M} -subobjects of X with $\text{colim } X_j \cong X$, we can send $\{X_j\}$ along (12) to a diagram in $(\mathcal{I}^\heartsuit)_1$, and deduce that X supports a structure of an object of $(\mathcal{I}^\heartsuit)_1$. Then, X is a fibrant object for $(\mathcal{L}, \mathcal{R})$. \square

7. PREORDER-ENRICHED AWFSS

We now turn our attention to the case of preorder-enriched AWFSS and their cofibrant KZ-generation. The case of a locally presentable base category can be easily extracted from the results in [4], so we here concentrate in the non-locally presentable case, as to include important examples as that of the category of topological spaces.

Assumption 28. Let us assume that our category $\mathcal{V} \subseteq \mathbf{Cat}$ is in fact a subcategory of the cartesian closed category \mathbf{Pord} of preorders.

Assumption 28 has an immediate simplifying consequence:

Lemma 29. *If $\mathbb{J} = (\mathcal{J} \rightrightarrows \mathcal{J}_0)$ is an internal category in $\mathcal{V}\text{-Cat}$ over $\text{Sq}(\mathcal{C})$, then the inclusion $\mathbb{J}^{\heartsuit_{\text{KZ}}} \hookrightarrow \mathcal{J}^{\heartsuit_{\text{KZ}}}$ is an identity.*

Proof. By definition, the inclusion is fully faithful and injective on objects. Denote by $U: \mathcal{J} \rightarrow \mathcal{C}^2$ the arrow part of the internal functor from \mathbb{J} to $\text{Sq}(\mathcal{C})$. Recall from Notation 12 that an object of $\mathcal{J}^{\heartsuit_{\text{KZ}}}$ is a morphism $f \in \mathcal{C}^2$ together with a RALI structure on $W_U(f)$, and that this object lies in $\mathbb{J}^{\heartsuit_{\text{KZ}}}$ if certain pair of RALI structures on $\mathcal{C}^2(Uj \cdot Ui, (1, f))$ coincide, for all pair of vertically composable $i, j \in \mathcal{J}$. In locally preordered 2-categories, RALI structures are unique, so all the objects of $\mathcal{J}^{\heartsuit_{\text{KZ}}}$ lie in $\mathbb{J}^{\heartsuit_{\text{KZ}}}$, concluding the proof. \square

Lemma 30. *Let M be the \mathcal{V} -monad whose algebras are RALIs. The forgetful \mathcal{V} -functor $M\text{-Alg} \rightarrow (M, \Lambda^M)\text{-Alg}$ is an isomorphism.*

Proof. The forgetful \mathcal{V} -functor in question is always full and faithful and injective on objects, so we only need to show it be surjective on objects. Algebras for the pointed endo-2-functor (M, Λ^M) has been already described in Remark 1 as a morphism $f: A \rightarrow B$ equipped with $f^\ell: B \rightarrow A$ and a 2-cell $\varepsilon: f^\ell \cdot f \leq 1_A$ such that $f \cdot f^\ell = 1_A$ and $f \cdot \varepsilon = 1$. In order to have an adjunction we only need $\varepsilon \cdot f = 1$, but $\varepsilon \cdot f$ has f as both its domain and codomain, and therefore is an identity because $\mathcal{V}(A, B)$ is a preorder. Then, (M, m) -algebras are the same as M -algebras, and the forgetful \mathcal{V} -functor is an isomorphism. \square

Theorem 31. *Let \mathcal{C} be a complete and cocomplete \mathcal{V} -enriched category whose objects have a rank with respect to a cocomplete OFS $(\mathcal{E}, \mathcal{M})$ on the underlying category \mathcal{C}_0 . Then the cofibrantly KZ-generated AWFSS on any internal category in $\mathcal{V}\text{-Cat}$ over $\text{Sq}(\mathcal{C})$ exists and is lax orthogonal.*

Proof. It suffices to consider the case of a small \mathcal{V} -category $U: \mathcal{J} \rightarrow \mathcal{C}^2$, by Lemma 29. By definition, $\mathcal{J}^{\heartsuit_{\text{KZ}}} \rightarrow \mathcal{C}^2$ is the pullback along W_U of $M\text{-Alg} \rightarrow [\mathcal{J}^{\text{op}}, \mathcal{V}]^2$, or equivalently, by Lemma 30, the pullback of $(M, \Lambda^M)\text{-Alg} \rightarrow [\mathcal{J}^{\text{op}}, \mathcal{V}]^2$. Since W_U has a left adjoint $L \dashv W_U$, then \mathcal{J}^\heartsuit is isomorphic to $(R, \Lambda^R)\text{-Alg}$ for the

pointed \mathcal{V} -functor (R, Λ^R) on \mathcal{C}^2 given by the pushout depicted on the right.

$$\begin{array}{ccc} \mathcal{J}^{\kappa_{\text{KZ}}} & \longrightarrow & (M, \Lambda^M)\text{-Alg} \\ \downarrow & & \downarrow \\ \mathcal{C}^2 & \xrightarrow{W_U} & [\mathcal{J}^{\text{op}}, \mathcal{V}]^2 \end{array} \quad \begin{array}{ccc} LW_U & \xrightarrow{\text{counit}} & 1 \\ L\Lambda^M W_U \downarrow & & \downarrow \Lambda^R \\ LMW_U & \longrightarrow & R \end{array}$$

To conclude the proof, it suffices to show that the functor

$$((R, \Lambda^R)\text{-Alg})_{\circ} = (R_{\circ}, \Lambda_{\circ}^R)\text{-Alg} \longrightarrow \mathcal{C}_{\circ}^2 \quad (13)$$

has a left adjoint. For, its left adjoint will underlie a left adjoint \mathcal{V} -functor, since $\mathcal{J}^{\kappa_{\text{KZ}}} \rightarrow \mathcal{C}^2$ preserves cotensor products with $\mathbf{2}$ – see proof of Lemma 4. It will then follow that $\mathcal{J}^{\kappa_{\text{KZ}}}$ is monadic over \mathcal{C}^2 , in the \mathcal{V} -enriched sense.

Now we prove that (13) has a left adjoint. First, we can show, in the same way as we did in the proof of Proposition 24, that W_U preserves κ -filtered colimits of \mathcal{M} -subobjects, for some κ . The next step in the proof is to observe that LW_U and LMW_U preserve any colimit that is preserved by W_U , since L and M are cocontinuous. Therefore, R preserves κ -filtered colimits of \mathcal{M} -subobjects. We may now use [15, 15.6] to deduce that (13) has a left adjoint, concluding the proof. \square

8. EXAMPLES OF NON-COFIBRANT GENERATION

In this section we exhibit two examples of lax idempotent AWFSS on \mathbf{a} that can be constructed via the “simple adjunction” method introduced in [6] but are *not* cofibrantly KZ-generated, nor cofibrantly generated. Furthermore, their underlying WFSS are not cofibrantly generated, in the usual sense of the term.

8.1. Example: complete lattices. It has been known for a long time that complete lattices can be characterised as the preorders that are KZ-injective or Kan injective to poset-embeddings [3, Prop. 1]. If we regard the category **Pord** of preorders as the 2-category of categories enriched in $\mathbf{2}$, complete lattices are the algebras for the 2-monad P on $\mathbf{2}\text{-Cat} = \mathbf{Pord}$ given by $X \mapsto [X^{\text{op}}, \mathbf{2}]$; this is the presheaf 2-monad, and $P(X)$ can be described as the poset of all down-closed subsets of X .

By results of [6], there is a lax orthogonal AWFS (L, R) on **Pord** whose L -coalgebras are the full and faithful $\mathbf{2}$ -functors, ie the morphisms of preorders that are full when regarded as functors (note that these need not be injective). Furthermore, the **Pord**-category $R_1\text{-Alg}$ of algebras for the restriction of R to the fibre over 1 is the category **CompL** of complete lattices and order-preserving functions that preserve arbitrary suprema. The functorial factorisation that underlies this AWFS can be described in the following terms. If $f: A \rightarrow B$ is a morphism of preorders, then consider

$$Kf = \{(\phi, b) \in P(A) \times B : \forall b' \in B ((\exists a \in A (b' \leq f(a))) \Rightarrow (b' \leq b))\}$$

with the preorder structure inherited from $P(A) \times B$, and the morphisms $\lambda_f: A \rightarrow Kf$ and $\rho_f: Kf \rightarrow B$ given by $\lambda_f(a) = (\{a' \in A : a' \leq a\}, f(a))$ and $\rho_f(\phi, b) = b$. Then $f \mapsto (\lambda_f, \rho_f)$ is the given **Pord**-functorial factorisation.

Due to the fact that the fibres of the R -algebras are posets, [6, Lemma 11.6] guarantees that, if we call $(\mathcal{L}, \mathcal{R})$ the underlying WFS of (L, R) , a morphism $f \in \mathcal{L}$ if and only if it admits an L -coalgebra structure; and $f \in \mathcal{R}$ if and only if it admits an R -algebra structure.

Remark 32. There is a WFS $(\text{Full}, \text{Top})$ on **Cat**, introduced in [1], whose left morphisms are the full functors and whose right morphisms are the topological functors. Its restriction of **Pord** is precisely the $(\mathcal{L}, \mathcal{R})$ we are discussing. Furthermore, [1]

proves that the restriction of $(\mathcal{L}, \mathcal{R})$ to the category of posets is not cofibrantly generated, from where it is easy to deduce that $(\mathcal{L}, \mathcal{R})$ is not cofibrantly generated either.

Theorem 33. *The AWFS on **Pord** described at the beginning of the section is not cofibrantly KZ-generated nor cofibrantly generated. Furthermore, its underlying WFS is not cofibrantly generated, in the usual sense of the term.*

Proof. In order to apply Corollary 25, equip the ordinary category of preorders with the OFS $(\mathcal{E}, \mathcal{M})$ with \mathcal{E} = epimorphisms = surjections, and \mathcal{M} = strong monomorphisms = embeddings. Example 34 now shows that the AWFS cannot be cofibrantly generated with respect to any cocontinuous **Pord**-enriched AWFS. The same example and Proposition 27 imply that $(\mathcal{L}, \mathcal{R})$ is not cofibrantly generated; one can also appeal to [1], as mentioned in Remark 32. \square

The example that follows describes a well known fact that, nonetheless, we find useful to make explicit in order fix the notation that we shall use in Example 37.

Example 34. In the next paragraphs we show that, for all limit ordinals β , the forgetful **Pord**-functor **CompL** \rightarrow **Pord** does not create β -filtered unions of sub-objects.

We use without further mention the ordering of ordinals by the relation \in ; this is, $\alpha < \beta$ means precisely $\alpha \in \beta$. In this way, an ordinal β is the completely ordered set of those ordinals $\alpha \in \beta$. A subset $D \subseteq \beta$ of an ordinal β is bounded if and only if it has a supremum; the supremum is given by $\sup D = \min\{\gamma \in \beta : \forall \delta (\delta \in D \Rightarrow \delta \leq \gamma)\}$. In fact, $\sup D$ is the union of $\{\delta : \delta \in D\}$; in other words, that $(\gamma \in \sup D)$ if and only if $(\exists \delta \in D : \gamma \in \delta)$. Therefore, an ordinal is a complete lattice if and only if every subset is bounded, if and only if it has a top element. It is easy to see that an ordinal with a top element must be a successor ordinal: if $\tau \in \alpha$ is the top element of a limit ordinal α , then $\tau \in \alpha \in \text{succ}(\tau)$, since $\text{succ}(\tau) \notin \alpha$; but $\text{succ}(\tau) = \tau \cup \{\tau\}$, from where it follows the contradictory statement that $\alpha = \tau$.

Given a limit ordinal β , consider the functor $\beta \rightarrow \mathbf{Pord}$ given by $\mu \mapsto \text{succ}(\mu) = \mu \cup \{\mu\}$. The cocone given by the inclusions $\text{succ}(\mu) \hookrightarrow \beta$ is a colimiting cocone in **Pord**. Furthermore, for $\mu < \nu$, the inclusion $\text{succ}(\mu) \subset \text{succ}(\nu)$ preserves arbitrary suprema (colimits).

When β is a regular cardinal, this colimit is β -filtered, and we have exhibited each regular cardinal β as a β -filtered colimit of sub-preorders that are complete lattices. The ordinal β , however, is a limit ordinal and thus not a complete lattice, concluding the example.

Remark 35. The last sentence of Theorem 33 was first proved in [1]. The other two parts of the theorem can be proved without appealing to Corollary 25 by the technique used in [4] for locally presentable categories. In our case, **Pord** is locally presentable. Suppose that the AWFS is cofibrantly generated with respect to an accessible AWFS (\mathbf{E}, \mathbf{M}) on \mathcal{V} . By the same argument of the proof of [4, Prop. 13] applied to the monad \mathbf{M} , instead of the monad for split epimorphisms, shows that (\mathbf{L}, \mathbf{R}) must be accessible. Thus, the forgetful $\mathbf{R}_1\text{-Alg} \rightarrow \mathbf{Pord}$ is accessible, and, being conservative, it creates κ -filtered colimits for some regular cardinal κ , contradicting Example 34.

8.2. Example: continuous lattices. In this section we give another example of an AWFS that can be constructed via simple adjunctions [6] but is not cofibrantly generated. First we consider the AWFS on the category of T_0 topological spaces and conclude with the case of general topological spaces.

Let \mathbf{Top}_0 be the category of T_0 topological spaces and continuous maps. Each T_0 space X carries a posetal structure defined by $x \sqsubseteq y$ if and only if each neighbourhood of x is also a neighbourhood of y ; equivalently, if $x \in \overline{\{y\}}$. This is sometimes called the specialisation order. A continuous function $f: X \rightarrow Y$ becomes an order-preserving function $f: (X, \sqsubseteq) \rightarrow (Y, \sqsubseteq)$, so we have a functor $\mathbf{Top}_0 \rightarrow \mathbf{Pos}$ into the category of posets. The cartesian closed category \mathbf{Pos} can play the role of the base of enrichment \mathcal{V} of the previous sections, and we can make \mathbf{Top}_0 into a \mathbf{Pos} -category by declaring $f \leq g$ if the associated morphisms of posets satisfy $f \leq g$. In elementary terms, $f \leq g$ if and only if $f(x) \sqsubseteq g(x)$ for all x .

We now recall the definition of continuous lattices [22]. Suppose that (L, \leq) is a complete poset. Given a pair of elements $x, y \in L$ we say that x is *way below* y , written $x \ll y$, if for all directed subsets $D \subseteq L$, if $y \leq \vee D$ then there exists some $d \in D$ with $x \leq d$. A *continuous lattice* is a complete poset where every element is the supremum of the elements way below it:

$$x = \bigvee \{y : y \ll x\}.$$

Equip L with the Scott topology τ_L , whose open sets are those subsets $U \subseteq L$ that satisfy: (1) if $x \in U$ and $x \leq y$, then $y \in U$; (2) if $D \subseteq L$ is a directed subset and $\bigvee D \in U$, then $D \cap U \neq \emptyset$. In this way we can regard any continuous lattice as a T_0 topological space, and the specialisation order for this topology coincides with the order of L . Conversely, if the poset (X, \sqsubseteq) of a T_0 space X is a continuous lattice, the topology $\tau_{(X, \sqsubseteq)}$ coincides with the original topology of X . In this way, continuous lattices can be identified with a class of T_0 topological spaces.

A function $f: L \rightarrow L'$ between continuous lattices is continuous for the associated topology if and only if it preserves directed suprema. Thus, the category of continuous lattices and maps of poset that preserve directed suprema is isomorphic to a full subcategory \mathbf{CL} of \mathbf{Top}_0 .

As part of the seminal work [22], D. Scott showed that a topological space is a continuous lattice if and only if it is injective with respect to all embeddings of topological spaces. Later, it was shown in [7] that the category of continuous lattices is isomorphic to the category of algebras of the filter monad F on \mathbf{Top}_0 , the monad that assigns to each space X its space of filters FX endowed with a suitable topology. This is in fact a monad enriched in \mathbf{Pos} . Using that F is lax idempotent, or Kock–Zöberlein, [8] proves that continuous lattices are the spaces Kan injective, or lax orthogonal, to embeddings.

There is a lax orthogonal factorisation system (L, R) on \mathbf{Top}_0 such that: (1) the fibre replacement \mathbf{Pos} -enriched monad on \mathbf{Top}_0 is the filter monad F ; (2) a continuous function f is an L -coalgebra if and only if f is an embedding. This AWFS was constructed via the method of pulling back along a “simple adjunction” in [6], and described only as a WFS in [5].

Theorem 36. *The AWFS on \mathbf{Top}_0 described is not cofibrantly KZ-generated nor cofibrantly generated. Its underlying WFS is not cofibrantly generated.*

In fact, we shall show that this AWFS is not cofibrantly generated with respect to any cocontinuous \mathbf{Pos} -enriched AWFS (E, M) on \mathbf{Pos} .

Proof of Theorem 36. We first tackle the part of the statement that deals with the AWFS. The proof is an application of Proposition 24. The category \mathcal{C} will be \mathbf{Top}_0 of T_0 topological spaces, regarded as a \mathbf{Pos} -category via the specialisation order. The OFS $(\mathcal{E}, \mathcal{M})$ on \mathbf{Top}_0 has \mathcal{E} the class of surjections and \mathcal{M} the class of injections that are a homeomorphism onto their image – extremal monomorphisms. It is well known that each topological space has a rank with respect to $(\mathcal{E}, \mathcal{M})$.

Suppose that the AWFS (L, R) described at the beginning of the section is cofibrantly generated with respect to a cocontinuous AWFS (E, M) on **Pos**. By Proposition 24, the forgetful functor $R\text{-Alg} \rightarrow \mathcal{C}^2$ creates κ -filtered unions of \mathcal{M} -subobjects, for some regular cardinal κ . In particular, if R_1 is the associated fibrant replacement monad, ie the restriction of R to $\mathbf{Top}_0/1 \cong \mathbf{Top}_0$, then the forgetful $\mathbf{CL} \cong R_1\text{-Alg} \rightarrow \mathbf{Top}_0$ creates the said colimits. This is a contradiction, as exhibited by the following example, which, furthermore, together with Proposition 27 shows that the underlying WFS of (L, R) cannot be cofibrantly generated. \square

Example 37. In the next few paragraphs, which should be read as a continuation of Example 34, we show that, for each regular cardinal β , there is a β -filtered colimit of continuous lattices that is *not* created by the forgetful functor $\mathbf{CL} \rightarrow \mathbf{Top}_0$. We continue with the notations of Example 34.

The way-below relation $\alpha \ll \nu$ on an ordinal β satisfies: if $\alpha \in \nu \in \beta$, then $\alpha \ll \nu$. For, if $D \subset \beta$ verifies $\nu \leq \sup D$, then $\alpha \in \sup D$ and there must be a $\delta \in D$ with $\alpha \in \delta$.

It is now easy to verify that successor ordinals are continuous lattices, as they are complete and any α is the supremum of those ordinals $\gamma \in \alpha$. If $\mu \in \nu$, the inclusion $\text{succ}(\mu) \subset \text{succ}(\nu)$ is continuous for the Scott topology, since it preserves suprema.

Now suppose that β is a limit ordinal. Then β is a filtered union of subspaces $\text{succ}(\mu) \subseteq \beta$, for $\mu \in \beta$. The cocone $\text{succ}(\mu) \hookrightarrow \beta$ exhibits β as a colimit of the $\text{succ}(\mu)$, and we can make this a colimit in \mathbf{Top}_0 by equipping β with the colimit topology induced by the Scott topology of the continuous lattices $\text{succ}(\mu)$: a subset $U \subseteq \beta$ is open if $U \cap \text{succ}(\mu)$ is open in $\text{succ}(\mu)$, for all $\mu \in \beta$. More explicitly, $U \subseteq \beta$ is open if: (1) it is up-closed, and; (2) for any bounded $D \subseteq \beta$, $\sup D \in U$ implies $U \cap D \neq \emptyset$. With this topology, $\text{succ}(\mu) \hookrightarrow \beta$ is an embedding of spaces.

Now assume further that β is a regular cardinal, so (β, \leq) is a β -filtered ordered set. By the above paragraph, β is a β -filtered colimit of continuous lattices. But β is not a continuous lattice, as it is not complete: it lacks a top element.

The AWFS (L, R) on \mathbf{Top}_0 of this section is the restriction of another on **Top**, for which we may use the same name; indeed, the whole of [6, §12] deals with the category **Top** of all topological spaces. The L-coalgebras of this AWFS are the continuous maps $f: X \rightarrow Y$ such that $f_*: \mathcal{O}(X) \rightarrow \mathcal{O}(Y)$ is a full morphism of posets, where $f_*(U) = \cup \{V \in \mathcal{O}(Y) : f^{-1}(V) \subseteq U\}$.

Corollary 38. *The AWFS (L, R) on **Top** is not cofibrantly KZ-generated, nor cofibrantly generated. Furthermore, its underlying WFS is not cofibrantly generated.*

Proof. If the AWFS were cofibrantly generated by a cocontinuous AWFS on **Pord**, then the right hand side vertical arrow in the following pullback square would create κ -filtered colimits of subspace embeddings, for some regular cardinal κ .

$$\begin{array}{ccc} \mathbf{CL} & \longrightarrow & R_1\text{-Alg} \\ \downarrow & & \downarrow \\ \mathbf{Top}_0 & \hookrightarrow & \mathbf{Top} \end{array}$$

As a consequence, the forgetful functor on the left hand side would also create κ -filtered colimits of embeddings, contradicting Example 37. Regarding the WFS on **Top** underlying (L, R) , its cofibrant objects are the continuous lattices; see [6, §12] for a full explanation. Then, Proposition 27 together with Example 37 show that the WFS cannot be cofibrantly generated. \square

APPENDIX A. COMPARISON WITH OTHER WORK

Adámek, Sousa and Velebil [2] studied conditions under which the objects of a poset-enriched category that are Kan injective with respect to a family of morphisms form a monadic category whose monad is lax idempotent. In this short appendix we compare our results with some of the contents of [2]. We begin by giving the context.

An object A of a locally poset-enriched category \mathcal{C} is *Kan injective* with respect to a morphism $j: X \rightarrow Y$ if each morphism $f: X \rightarrow A$ has a left Kan extension $\text{Lan}_j f$ along f , and $(\text{Lan}_j f) \cdot j = f$. This left Kan extension is unique if it exists, as the only invertible 2-cells are identities; the notion of Kan injectivity already appears in [9]. – Note the difference with the case of preorder-enriched categories, where these Kan extensions need not be unique. – If \mathcal{J} is a class of morphisms, one says that A is Kan injective with respect to \mathcal{J} if it is Kan injective with respect to each member of \mathcal{J} . There is a sub-2-category $\text{Llnj}(\mathcal{J})$ of \mathcal{C} whose objects are those which are Kan injective with respect to \mathcal{J} , and whose morphisms are those which preserve all the left Kan extensions along elements of \mathcal{J} . Due to the uniqueness of Kan extensions, the existence of $\text{Lan}_j f$ equates the choice of one such Kan extension, so the poset-enriched category $\text{Llnj}(\mathcal{J})$ is isomorphic to the fibre of the codomain functor $\mathcal{J}^{\text{rkz}} \rightarrow \mathcal{C}^2 \rightarrow \mathcal{C}$ over the terminal object, if a terminal object exists.

In the terminology of [2], a poset-enriched category \mathcal{C} is locally ranked if it is cocomplete as a category enriched in posets, it has a proper orthogonal factorisation system $(\mathcal{E}, \mathcal{M})$ where the elements of \mathcal{M} are order-monomorphisms, and \mathcal{C}_o is cowellpowered with respect to $(\mathcal{E}, \mathcal{M})$ and each object has a rank.

The main result of [2] is its Theorem 6.10, which states that, if \mathcal{C} is locally ranked, the inclusion of $\text{Llnj}(\mathcal{J})$ into \mathcal{C} has a left adjoint whenever the elements of \mathcal{J} that are not order-epimorphisms form a small set. The main component of the proof is the construction of the Kan-injective reflection chain in [2, §5].

Now that we have outlined the ingredients of the Kan injectivity of Adámek, Sousa and Velebil, we can proceed to compare it with our results. First, we are able to work with preorder-enriched categories instead of poset-enriched ones; this allows examples such as the category of topological spaces, Lawvere metric spaces [19] and others. Furthermore, we can deal with injectivity with respect to a preorder-enriched functor $\mathcal{J} \rightarrow \mathcal{C}^2$ instead of a family of morphisms in \mathcal{C} . When \mathcal{J} is discrete, we recover the case of a family of morphisms. Our Theorem 21 generalises [2, Cor. 4.12] when \mathcal{C} is complete and cocomplete.

If \mathcal{J} is a class of morphisms in a poset-enriched category \mathcal{C} with a terminal object, the poset-enriched category $\text{Llnj}(\mathcal{J})$ is, as already mentioned, the fibre over the terminal object of the codomain functor $\mathcal{J}^{\text{rkz}} \rightarrow \mathcal{C}$. Theorem 31 deals with the monadicity of \mathcal{J}^{rkz} over \mathcal{C}^2 , and by restricting to the fibre over the terminal object, one can deduce the 2-monadicity of $\text{Llnj}(\mathcal{J})$ over \mathcal{C} , yielding [2, Thm. 6.10] – but not the explicit construction in its proof – as long as there is a terminal object. Furthermore, our hypotheses on the OFS $(\mathcal{E}, \mathcal{M})$ are slightly weaker.

A difference of generality in the opposite direction is that the family of morphisms in [2, Thm. 6.10] has to be small only up to a possibly large family of order-epimorphisms, while the preorder-enriched category \mathcal{J} of our theorem is required to be small.

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